Worked Solutions

Question 1

The plant is

$$G(s) = \frac{1}{s(s+1)(s+2)}, \qquad L(s) = kG(s),$$

with unity negative feedback.

1(a)(i) Nyquist sketch and stable range of k. 25%

Open-loop poles: $s = 0, -1, -2$. There are no open-loop RHP poles, but there is a pole on the imaginary axis at the origin, so the Nyquist D-contour must be indented around $s = 0$.

For $\omega > 0$,

$$G(j\omega) = \frac{1}{j\omega(1 + j\omega)(2 + j\omega)}.$$

Asymptotes:

$$\omega \to 0^+ : \quad G(j\omega) \sim \frac{1}{2j\omega} = -\frac{j}{2\omega} \;\to\; -j\infty,$$

$$\omega \to \infty : \quad G(j\omega) \sim \frac{1}{(j\omega)^3} = \frac{j}{\omega^3} \;\to\; 0 \text{ along } +j.$$

Real-axis crossing: $\mathrm{Im}\{G(j\omega)\} = 0$ occurs when the overall phase is -180° (mod 180°). One convenient way is to note that

$$\mathrm{Im}\{G(j\omega)\} = 0 \quad \Longleftrightarrow \quad \omega = 0 \text{ or } \omega = \sqrt{2}.$$

At $\omega = \sqrt{2}$,

$$G(j\sqrt{2}) = -\frac{1}{6}.$$

By symmetry, the $\omega < 0$ part is the complex conjugate of the $\omega > 0$ part, so the Nyquist plot is symmetric about the real axis. The indentation around $s = 0$ maps to a large arc at infinity in the G-plane, closing the contour.

Nyquist criterion: with $P = 0$ open-loop RHP poles, closed-loop stability requires $N = 0$ net clockwise encirclements of -1 by the Nyquist plot of $L(s) = kG(s)$.

Scaling by k moves the negative-real crossing from $-1/6$ to $-k/6$. The critical value is when this equals -1:

$$-\frac{k}{6} = -1 \quad \Rightarrow \quad k = 6.$$

Hence

$$\boxed{\text{The closed loop is internally stable for } 0 < k < 6.}$$

At $k = 6$ the plot passes through -1 (marginal stability); for $k > 6$ the plot encircles -1 giving instability. For $k < 0$ the closed-loop polynomial has a positive real root, so the closed loop is unstable.

1(a)(ii) Delay margin for $k = 2$. 20%

With delay, the return ratio is

$$L(s) = kG(s)e^{-sT}, \qquad k = 2.$$

A pure delay does not change the magnitude but adds phase $-\omega T$.

Gain crossover. The gain-crossover frequency ω_c satisfies

$$|L(j\omega_c)| = 1 \iff k|G(j\omega_c)| = 1.$$

Since

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}\sqrt{4+\omega^2}},$$

the crossover solves

$$\frac{2}{\omega_c\sqrt{1+\omega_c^2}\sqrt{4+\omega_c^2}} = 1 \iff \omega_c\sqrt{1+\omega_c^2}\sqrt{4+\omega_c^2} = 2.$$

For $\omega_c < 1$, we get $\omega_c^2(1+\omega_c)^2 \approx 1$ and $\omega_c \approx \sqrt{0.618} \approx 0.79$, or solve numerically to give

$$\boxed{\omega_c \approx 0.749 \text{ rad/s.}}$$

Phase margin (no delay). The phase of $G(j\omega)$ is

$$\angle G(j\omega) = -90^\circ - \arctan(\omega) - \arctan(\omega/2).$$

At $\omega_c \approx 0.749$,

$$\angle G(j\omega_c) \approx -90^\circ - 36.9^\circ - 20.5^\circ \approx -147.4^\circ.$$

Therefore the phase margin of $L(s) = 2G(s)$ at crossover is

$$\text{PM} \approx 180^\circ - 147.4^\circ \approx 32.6^\circ \approx 0.569 \text{ rad.}$$

Delay margin.

$$\omega_c T_{\max} \approx \text{PM} \implies \boxed{T_{\max} \approx \frac{0.569}{0.749} \approx 0.76 \text{ s.}}$$

1(b)(i) Small-gain sufficient condition. 20%

Nominal plant:

$$G_0(s) = \frac{1}{2s(s+1)}, \qquad L_0(s) = kG_0(s).$$

Uncertainty set:

$$G_\Delta(s) = G_0(s)(1+\Delta(s)), \qquad \Delta \text{ stable}, \qquad |\Delta(j\omega)| \le \frac{\omega}{2}.$$

The perturbed loop transfer is $\hat{L}(s) = L_0(s)(1+\Delta(s))$, so

$$1 + \hat{L} = 1 + L_0 + L_0\Delta = (1+L_0)\Big(1 + \Delta\,\frac{L_0}{1+L_0}\Big) = (1+L_0)\big(1+\Delta T_0\big),$$

where the nominal complementary sensitivity is

$$T_0(s) = \frac{L_0(s)}{1+L_0(s)}.$$

Assuming the nominal closed loop is internally stable, robust internal stability reduces to stability of the feedback interconnection of the stable blocks Δ and T_0. By the small-gain theorem, a sufficient condition is

$$\sup_{\omega\geq 0} |\Delta(j\omega)T_0(j\omega)| < 1.$$

Using the bound $|\Delta(j\omega)| \leq \omega/2$ gives the convenient sufficient condition

$$\boxed{\sup_{\omega\geq 0} \frac{\omega}{2} |T_0(j\omega)| < 1.}$$

1(b)(ii) Explicit range of k 25%

First write T_0 explicitly:

$$L_0(s) = \frac{k}{2s(s+1)} \quad \Rightarrow \quad T_0(s) = \frac{L_0}{1+L_0} = \frac{k}{2s(s+1)+k} = \frac{k/2}{s^2+s+k/2}.$$

On the imaginary axis, $s = j\omega$:

$$T_0(j\omega) = \frac{k/2}{(k/2-\omega^2)+j\omega}, \qquad |T_0(j\omega)| = \frac{k/2}{\sqrt{(k/2-\omega^2)^2+\omega^2}}.$$

Thus

$$\frac{\omega}{2}|T_0(j\omega)| = \frac{k\omega}{4\sqrt{(k/2-\omega^2)^2+\omega^2}}.$$

Maximising the square of this over ω^2 one finds the maximum occurs at $\omega^2 = k/2$ giving

$$\sup_{\omega\geq 0} \frac{\omega}{2}|T_0(j\omega)| = \frac{k}{4}.$$

Therefore the sufficient robust-stability condition becomes

$$\frac{k}{4} < 1 \quad \Longleftrightarrow \quad \boxed{0 < k < 4}$$

1(b)(iii) Comparison with part (a). 10%

First Note that $G_\Delta = G$ for $\Delta = -s/(s+2)$, in which case $|\Delta(j\omega)| \leq \omega/2$.

Part (a) (exact third-order plant) gives stability for

$$0 < k < 6.$$

Part (b) gives the *robust* small-gain guarantee

$$0 < k < 4,$$

which is narrower because (i) we demand stability for a whole uncertainty set (including additional unmodelled dynamics), and (ii) the small-gain test is sufficient and therefore potentially conservative.

Question 2

Let $L(s) = G(s)K(s)$ and $T(s) = \dfrac{L(s)}{1+L(s)}$, where

$$G(s) = \frac{s^2+0.2s+1}{(s^2+0.2s+9)(s-2)}.$$

2(a)(i) Why internal stability implies $T(2) = 1$. 15%

The plant has an unstable pole at $s = 2$ (due to the factor $(s - 2)$ in the denominator). Under internal stability we exclude unstable pole–zero cancellation, so we may assume $K(s)$ has no zero at $s = 2$, and hence $L(s) = G(s)K(s)$ has a pole at $s = 2$.

Write, near $s = 2$,

$$L(s) = \frac{\ell(s)}{s-2},$$

where $\ell(s)$ is analytic and $\ell(2) \neq 0$. Then

$$T(s) = \frac{L(s)}{1+L(s)} = \frac{\ell(s)/(s-2)}{1+\ell(s)/(s-2)} = \frac{\ell(s)}{(s-2)+\ell(s)}.$$

Taking $s \to 2$ gives

$$T(2) = \frac{\ell(2)}{\ell(2)} = 1.$$

Hence

$$\boxed{T(2) = 1.}$$

2(a)(ii) Poisson integral identity. 20%

Assume $T(s)$ is analytic and has no zeros in the open RHP (minimum phase) and decays sufficiently at infinity so that the Poisson integral formula applies. Using the formula sheet Poisson integral with $\sigma_0 = 2$ and $\omega_0 = 0$:

$$\int_0^\infty \frac{\sigma_0}{\sigma_0^2+\omega^2} \ln|T(j\omega)|\, d\omega = \frac{\pi}{2} \ln|T(\sigma_0)|.$$

With $\sigma_0 = 2$ this becomes

$$\int_0^\infty \frac{2}{4+\omega^2} \ln|T(j\omega)|\, d\omega = \frac{\pi}{2} \ln|T(2)|.$$

Using $T(2) = 1$ from part (i) gives $\ln|T(2)| = 0$, so the right-hand side is zero. Multiplying through by 2 yields

$$\boxed{\int_0^\infty \frac{4}{4+\omega^2} \ln|T(j\omega)|\, d\omega = 0.}$$

2(a)(iii) Waterbed interpretation. 15%

The identity is a *weighted average* constraint on $\ln|T(j\omega)|$:

$$\int_0^\infty w(\omega) \ln|T(j\omega)|\, d\omega = 0, \qquad w(\omega) = \frac{4}{4+\omega^2} > 0.$$

If $|T(j\omega)| < 1$ over a frequency band, then $\ln|T(j\omega)| < 0$ on that band and contributes *negative area*. To keep the weighted integral equal to zero, there must be *positive area* elsewhere, i.e. there must exist some frequencies for which $\ln|T(j\omega)| > 0$ or equivalently

$$|T(j\omega)| > 1.$$

This is the "waterbed effect": pushing $|T|$ down in one region forces it to rise above unity elsewhere. Note that asking for $|T|$ small (equivalently $|L|$ small) for frequencies < 2 is particularly bad, as w is large there, so any sensor noise below this frequency is particularly bad.

2(b)(i) Root-locus sketch (qualitative features). 20%

With proportional feedback $K(s) = k$ $(k > 0)$, the open-loop transfer is $L(s) = kG(s)$.

Open-loop poles:

$$s = 2, \qquad s = \frac{-0.2 \pm j\sqrt{35.96}}{2} \approx -0.1 \pm j2.998.$$

Open-loop zeros:

$$s = \frac{-0.2 \pm j\sqrt{3.96}}{2} \approx -0.1 \pm j0.995.$$

Since there are $n = 3$ poles and $m = 2$ zeros, there is one asymptote with angle

$$\theta = \frac{(2q+1)180^\circ}{n - m} = 180^\circ,$$

i.e. the remaining branch goes to $-\infty$ along the negative real axis.

There is a real-axis locus on $(-\infty, 2)$ (to the left of an odd number of real poles/zeros; here the only real pole is at 2). One branch starts at $s = 2$ and moves left along the real axis to $-\infty$. The remaining two branches start at the complex pole pair and terminate at the complex zero pair, moving symmetrically about the real axis and (for some gain range) intersecting the real axis near the given break points.

A sketch should show: one real branch from 2 to $-\infty$, and a complex-conjugate pair travelling from $-0.1 \pm j2.998$ to $-0.1 \pm j0.995$.

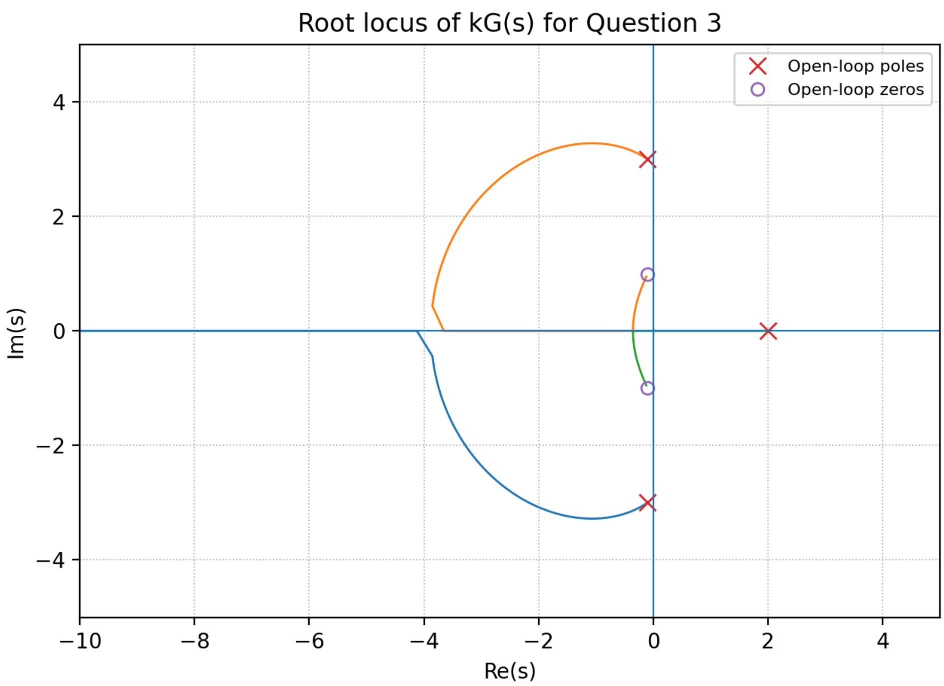

Figure 1: Computed root locus of $kG(s)$ for Question 2(b) (open-loop poles marked $\times$, open-loop zeros marked $\circ$).

2(b)(ii) Gain k that places poles furthest left. 20%

For a point s_0 on the root locus with $k > 0$,

$$1 + kG(s_0) = 0 \quad \Rightarrow \quad k = -\frac{1}{G(s_0)}.$$

Using the given breakaway points:

- At $s = -0.35$,

$$k = -\frac{1}{G(-0.35)} \approx 20.21.$$

- At $s = -3.88$,

$$k = -\frac{1}{G(-3.88)} \approx 8.96.$$

However, at $k \approx 8.96$ the closed-loop still has one pole in the RHP (the system is not stable). The gain that makes the *rightmost* closed-loop pole as far left as possible while maintaining stability occurs at the other break point, $s \approx -0.25$, where the dominant poles coalesce on the real axis before moving back toward the zeros at $-0.1 \pm j0.995$ as k increases further.

Therefore

$$\boxed{k \approx 20.2.}$$

2(b)(iii) Two-degree-of-freedom design. 10%

Use a standard 2DOF structure with a stable prefilter $H(s)$ on the reference:

$$e(s) = H(s)r(s) - y(s), \qquad u(s) = k\,e(s), \qquad y(s) = G(s)u(s).$$

Then the reference-to-output transfer function is

$$\frac{Y(s)}{R(s)} = H(s)\,\frac{kG(s)}{1 + kG(s)} = H(s)\,T(s).$$

We desire

$$\frac{Y(s)}{R(s)} = \frac{9}{(s+3)^2}.$$

Hence choose

$$H(s) = \frac{\dfrac{9}{(s+3)^2}}{T(s)} = \frac{9}{(s+3)^2}\,\frac{1 + kG(s)}{kG(s)}.$$

Writing $G(s) = N(s)/D(s)$ with

$$N(s) = s^2 + 0.2s + 1, \qquad D(s) = (s^2 + 0.2s + 9)(s - 2),$$

gives

$$T(s) = \frac{kN(s)}{D(s) + kN(s)}.$$

Therefore

$$\boxed{H(s) = \frac{9\big(D(s) + kN(s)\big)}{k\,N(s)\,(s+3)^2} = \frac{9}{k}\,\frac{s^3 + (k-1.8)s^2 + (8.6 + 0.2k)s + (k - 18)}{(s+3)^2\,(s^2 + 0.2s + 1)}.}$$

For any stabilising choice of k (e.g. $k \approx 20.2$ from part (ii)), $H(s)$ is proper (degree 3 over degree 4) and has only LHP poles (the poles of $N(s)$ and $(s+3)^2$), so the overall closed-loop remains internally stable and the command response matches the specification.

Question 3

(These parts correspond to the Bode plot in Fig. 1.)

3(a) Estimate $G_{\min}(s)$ from the gain plot. 25%

An all-pass factor $B(s)$ satisfies $|B(j\omega)| = 1$, so the *gain plot* determines $G_{\min}$.
From the magnitude plot:

- DC gain is about 40 dB $\Rightarrow |G(0)| \approx 10^{40/20} = 100$.
- High-frequency roll-off tends to about -60 dB/dec $\Rightarrow$ relative degree ≈ 3 (three more poles than zeros).
- Three "knees" suggest three poles at roughly

$$\omega_{p1} \approx 0.2 \text{ rad/s}, \quad \omega_{p2} \approx 5 \text{ rad/s}, \quad \omega_{p3} \approx 50 \text{ rad/s},$$

  with no clear magnitude evidence of zeros.

A plausible minimum-phase model is therefore

$$\boxed{G_{\min}(s) \approx \frac{100}{(1+s/0.2)(1+s/5)(1+s/50)}}.$$

3(b) Sketch $\angle G_{\min}(j\omega)$. 15%

For three well-separated poles, a straight-line phase approximation:

- $\omega \ll 0.2$: phase $\approx 0°$,
- around $\omega \approx 0.2$: phase transitions toward $-90°$,
- around $\omega \approx 5$: toward $-180°$,
- around $\omega \approx 50$: toward $-270°$,
- $\omega \gg 50$: phase $\approx -270°$.

So you would draw a monotone decrease approaching $-270°$ at high frequency.

3(c) Infer all-pass factor(s) $B(s)$. 25%

The *measured* phase in Fig. 1 tends to about $-90°$ at high frequency, whereas the minimum-phase three-pole model tends to about $-270°$. The discrepancy is therefore approximately

$$(-90°) - (-270°) = +180°.$$

An all-pass factor that contributes an additional $+180°$ of phase (with no magnitude change) is an *unstable* first-order all-pass factor with a right-half-plane pole:

$$B(s) = \frac{s+a}{s-a}, \qquad a > 0,$$

for which $|B(j\omega)| = 1$ and $\angle B(j\omega)$ rises from $0°$ to $+180°$ as ω increases.

From the phase plot, the "midpoint" of this phase lift occurs around a few rad/s, suggesting a of order 1–3. A consistent estimate is $a \approx 2$ rad/s:

$$\boxed{B(s) \approx \frac{s+2}{s-2}},$$

indicating a right-half-plane pole near $s = +2$ (and a corresponding LHP zero at $s = -2$).

3(d) PI–lead loop shaping (one coherent choice, based on the full G. 35%)

Use

$$K(s) = k\Big(1 + \frac{\omega_I}{s}\Big)\frac{1 + s/\omega_z}{1 + s/\omega_p}, \qquad \omega_p > \omega_z > 0.$$

From parts (a)–(c) we may write (one consistent underlying model)

$$G(s) = G_{\min}(s)\, B(s) \approx \frac{5000}{(s+0.2)(s+5)(s+50)}\,\Big(-\frac{s+2}{s-2}\Big),$$

i.e. the plant has a right-half-plane pole near +2 and its phase in the desired crossover band (5–15 rad/s) is close to 0°.

Key design observation (sign of k). At $\omega \approx 10$ rad/s the plant phase is only a few degrees negative (see Fig. 1), whereas a classical "≈ 50°" phase margin at gain crossover requires the loop phase to be about −130°. The PI term is designed so that it contributes little phase at crossover, and the lead term contributes *positive* phase. Therefore an additional 180° phase inversion is needed in the loop; equivalently, we take $k < 0$.

One consistent numerical choice. Choose a gain crossover near $\omega_c \approx 10$ rad/s (within the required 5–15 rad/s range), put the PI zero one decade below crossover, and tune the lead so that its maximum phase advance occurs at ω_c:

$$\omega_I = 1 \text{ rad/s}, \qquad \sqrt{\omega_z\omega_p} = \omega_c = 10, \qquad \alpha := \frac{\omega_p}{\omega_z} = 16 \;\Rightarrow\; \omega_z = \frac{10}{\sqrt{16}} = 2.5, \;\; \omega_p = 10\sqrt{16} = 40.$$

For $\alpha = 16$ the maximum phase lead is

$$\phi_{\max} = \sin^{-1}\Big(\frac{\alpha-1}{\alpha+1}\Big) = \sin^{-1}\Big(\frac{15}{17}\Big) \approx 61.9°,$$

and (by construction) this occurs at $\omega = \omega_c$. The PI factor contributes a small lag at crossover:

$$\angle\Big(1 + \frac{\omega_I}{j\omega_c}\Big) = -\tan^{-1}\Big(\frac{\omega_I}{\omega_c}\Big) \approx -5.7°.$$

Hence the net controller phase at crossover is about $61.9° - 5.7° \approx 56°$. With $k < 0$, the extra −180° shift gives a loop phase near −130° at crossover, corresponding to a phase margin close to 50°.

To set $|L(j\omega_c)| \approx 1$, note that the lead magnitude at ω_c is $|K_{\text{lead}}(j\omega_c)| = \sqrt{\alpha} = 4$ and the PI magnitude is close to 1 at crossover. From the plant magnitude in Fig. 1, $|G(j10)| \approx 0.88$, so take

$$|k| \approx \frac{1}{|G(j10)|\,|K_{\text{PI}}(j10)|\,|K_{\text{lead}}(j10)|} \approx \frac{1}{0.88 \times 1 \times 4} \approx 0.284,$$

with negative sign:

$$\boxed{k \approx -0.284, \qquad \omega_I = 1 \text{ rad/s}, \qquad \omega_z = 2.5 \text{ rad/s}, \qquad \omega_p = 40 \text{ rad/s}.}$$

Bode diagram of the final return ratio. With the full plant $G(s)$ (including the all-pass/RHP-pole factor) and the controller above, the return ratio

$$L(s) = K(s)G(s)$$

has gain crossover at $\omega_c \approx 10$ rad/s and phase margin ≈ 50°. Moreover, for $\omega \geq 10^3$ rad/s the plant magnitude is extremely small, so $|L| \ll 1$ and hence $|T| \approx |L| \ll 1$ (good sensor-noise attenuation).

The Bode magnitude and phase of the final $L(j\omega)$ are shown in Figs. 2–3.

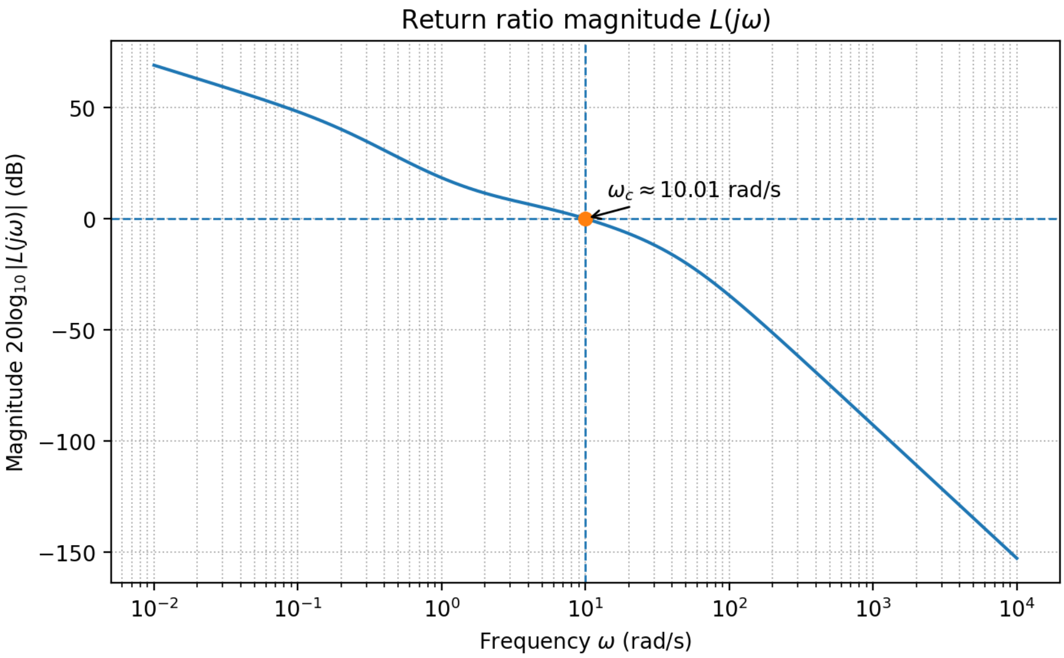

Figure 2: Bode magnitude of the final return ratio $L(j\omega)$ for Question 3(d) (using the full G). The gain crossover ω_c is marked.

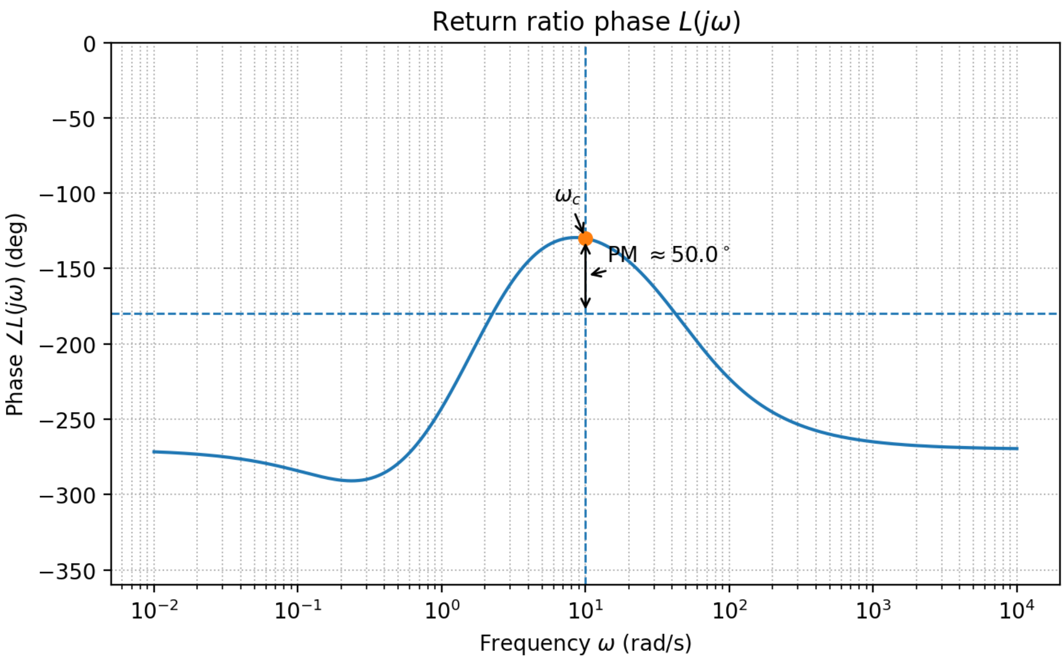

Figure 3: Bode phase of the final return ratio $L(j\omega)$ for Question 3(d) (using the full G). The phase at ω_c and the phase margin are marked.