EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 28 April 2026 14.00 to 15.40

Module 4F1

CONTROL SYSTEM DESIGN

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4F1 Formulae Sheet (3 pages)
Supplementary page: one extra copy of Fig.1 (Question 3)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 Consider the standard unity negative-feedback system with return ratio

$$L(s) = kG(s), \qquad G(s) = \frac{1}{s(s+1)(s+2)},$$

where k is a real constant.

(a) (i) Sketch the full Nyquist diagram of $G(s)$, paying attention to any required indentation of the Nyquist contour. Hence determine the range of k for which the closed-loop system is internally stable. [25%]

(ii) A pure time delay e^{-sT} is inserted in series with the plant so that $L(s) = kG(s)e^{-sT}$. For the gain $k = 2$, estimate the largest delay $T_{\max}$ for which the closed loop remains stable for $T < T_{\max}$. [20%]

(b) Define a nominal plant as

$$G_0(s) = \frac{1}{2s(s+1)}.$$

The true plant is assumed to lie in the multiplicative uncertainty set

$$G_\Delta(s) = G_0(s)\big(1 + \Delta(s)\big),$$

where $\Delta(s)$ is stable and satisfies

$$|\Delta(j\omega)| \leq \frac{\omega}{2} \qquad \text{for all } \omega \geq 0.$$

As before, the controller is $K(s) = k$.

(i) Using a small-gain argument, derive a sufficient condition for robust internal stability in terms of the nominal complementary sensitivity $T_0(s)$. [20%]

(ii) Hence determine an explicit range of k which ensures robust internal stability. *Hint: You might find it helpful to first find the maximum of a certain frequency response function over ω^2 for fixed k.* [25%]

(iii) Show that this uncertainty set includes the original $G(s)$ and briefly comment on how your results relate to those in part (a). [10%]

2 Consider an internally stable single-loop feedback system with return ratio $L(s) = G(s)K(s)$ and complementary sensitivity

$$T(s) = \frac{L(s)}{1 + L(s)}.$$

The plant transfer function is given by

$$G(s) = \frac{s^2 + 0.2s + 1}{(s^2 + 0.2s + 9)(s - 2)}.$$

(a) (i) Explain why internal stability implies $T(2) = 1$. [15%]

(ii) Using the Poisson integral, show that

$$\int_0^\infty \frac{4}{4 + \omega^2} \ln |T(j\omega)| \, d\omega = 0.$$

[20%]

(iii) Interpret this result in terms of the waterbed effect. Your answer should refer to the effect of noise sources in particular frequency ranges. [15%]

(b) (i) Show that breakaway points lie at $s = -0.35$ and $s = -3.88$ and sketch the root locus for $K(s) = k$, for all values of k. [20%]

(ii) Determine the value of k that places the closed-loop poles furthest to the left. [20%]

(iii) Design a two-degree-of-freedom control scheme to achieve

$$\frac{Y(s)}{R(s)} = \frac{9}{(s + 3)^2},$$

for a reference input $R(s)$ and plant output $Y(s)$, while maintaining internal stability.

[10%]

3 Figure 1 shows the Bode diagram of a transfer function $G(s)$. You may assume: (i) $G(s)$ is real-rational, (ii) all finite poles and zeros are simple and well separated, and (iii) $G(0) > 0$.

(a) From the gain plot, estimate a plausible $G_{\min}(s)$ for the minimum-phase factorisation

$$G(s) = G_{\min}(s)B(s),$$

where $B(s)$ is all-pass. Estimate the number of poles and zeros of $G_{\min}(s)$ and their approximate break frequencies. [25%]

(b) Using your answer to part (a), sketch the phase of $G_{\min}(j\omega)$ on the additional copy of the phase plot of Fig. 1. [15%]

(c) Hence infer the presence of any all-pass factor(s), indicating whether they correspond to right-half-plane poles or zeros. [25%]

(d) The plant is to be stabilised using a compensator of the form

$$K(s) = k\left(1 + \frac{\omega_I}{s}\right)\frac{1 + s/\omega_z}{1 + s/\omega_p}, \qquad \omega_p > \omega_z > 0.$$

Choose numerical values for the parameters so that the feedback system is stable, the loop has gain crossover no higher than 10 rad/s, a phase margin of about 50°, and good high-frequency noise attenuation. Explain your reasoning and why it is difficult to obtain a gain crossover of much less than 10 rad/s. [35%]

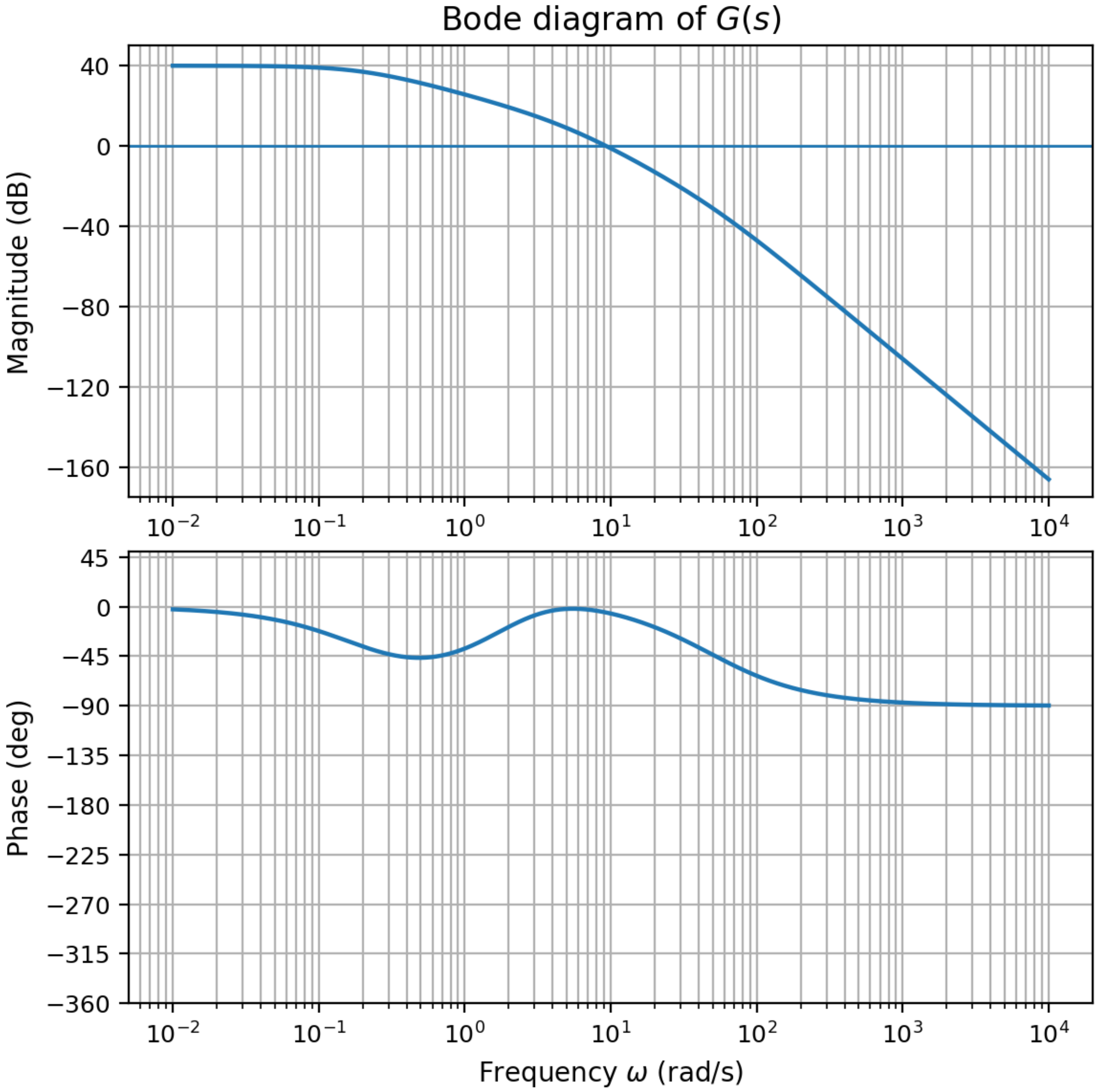

Fig. 1

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Formulae sheet for Module 4F1: Control System Design

To be available during the examination.

1 Terms

For the standard feedback system shown below, the **Return-Ratio Transfer Function** $L(s)$ is given by

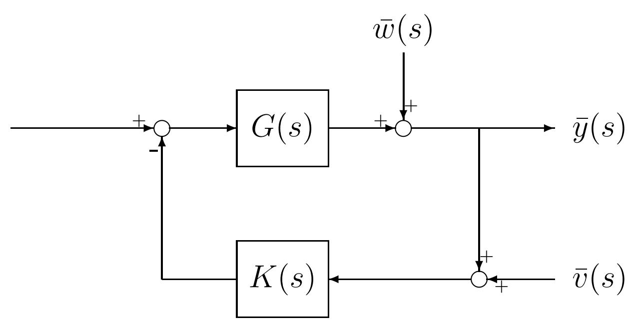
$$L(s) = G(s)K(s),$$

the **Sensitivity Function** $S(s)$ is given by

$$S(s) = \frac{1}{1 + G(s)K(s)}$$

and the **Complementary Sensitivity Function** $T(s)$ is given by

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$



The closed-loop system is called **Internally Stable** if each of the *four* closed-loop transfer functions

$$\frac{1}{1 + G(s)K(s)}, \quad \frac{G(s)K(s)}{1 + G(s)K(s)}, \quad \frac{K(s)}{1 + G(s)K(s)}, \quad \frac{G(s)}{1 + G(s)K(s)}$$

are stable (which is equivalent to $S(s)$ being stable and there being no right half plane pole/zero cancellations between $G(s)$ and $K(s)$).
A transfer function is called **real-rational** if it can be written as the ratio of two polynomials in s, the coefficients of each of which are purely real.

2 Phase-lead compensators

The phase-lead compensator

$$K(s) = \alpha \frac{s + \omega_c/\alpha}{s + \omega_c \alpha}, \quad \alpha > 1$$

achieves its maximum phase advance at $\omega = \omega_c$, and satisfies:

$$|K(j\omega_c)| = 1, \quad \text{and} \quad \angle K(j\omega_c) = 2 \arctan \alpha - 90°.$$

3 The Bode Gain/Phase Relationship

If

1. $L(s)$ is a real-rational function of s,
2. $L(s)$ has no poles or zeros in the *open* RHP ($\mathrm{Re}(s) > 0$) and
3. satisfies the normalization condition $L(0) > 0$.

then

$$\boxed{\angle L(j\omega_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d}{dv} \log |L(j\omega_0 e^v)| \log \coth \frac{|v|}{2}\, dv}$$

Note that

$$\log \coth \frac{|v|}{2} = \log \left| \frac{\omega + \omega_0}{\omega - \omega_0} \right|, \text{ where } \omega = \omega_0 e^v.$$

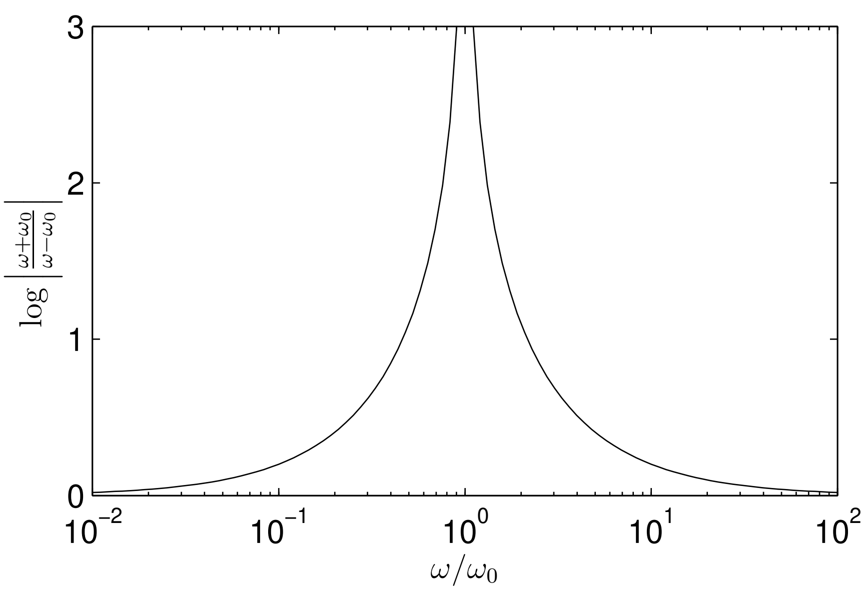


Figure 1:

If the slope of $L(j\omega)$ is approximately constant for a sufficiently wide range of frequencies around $\omega = \omega_0$ we get the *approximate form of the Bode Gain/Phase Relationship*

$$\angle L(j\omega_0) \approx \frac{\pi}{2} \left. \frac{d \log |L(j\omega_0 e^v)|}{dv} \right|_{v=0}.$$

4 The Poisson Integral

If $H(s)$ is a real-rational function of s which has no poles or zeros in $\mathrm{Re}(s) > 0$, then if $s_0 = \sigma_0 + j\omega_0$ with $\sigma_0 > 0$

$$\boxed{\log H(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} \log H(j\omega)\, d\omega}$$

and

$$\boxed{\log |H(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} \log \left|H\big(j|s_0|e^v\big)\right| dv}$$

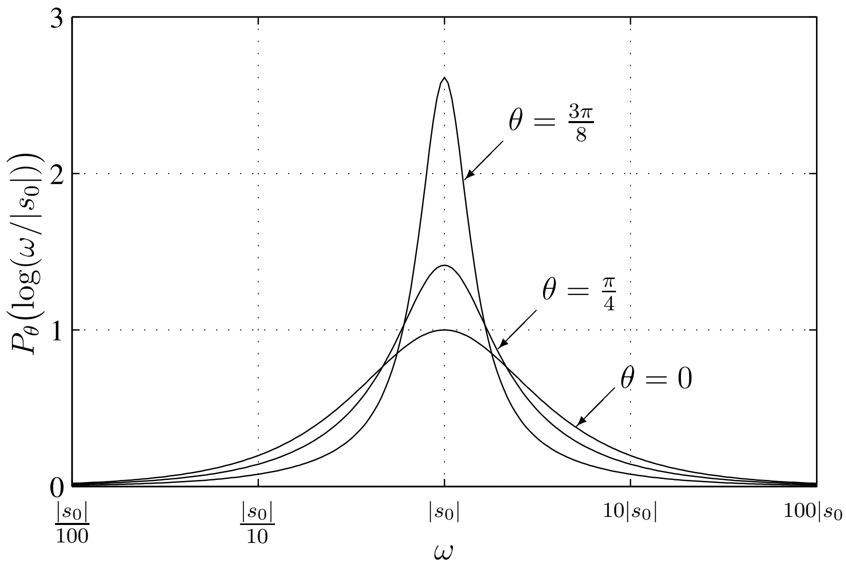
where $v = \log\left(\frac{\omega}{|s_0|}\right)$ and $\theta = \angle(s_0)$. Note that, if s_0 is real, so $\angle s_0 = 0$, then

$$\frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} = \frac{1}{\cosh v}.$$

We define

$$P_\theta(v) = \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta}$$

and give graphs of P_θ below.



The indefinite integral is given by

$$\int P_\theta(v)\, dv = \arctan\left(\frac{\sinh v}{\cos\theta}\right)$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} P_\theta(v)\, dv = 1 \quad \text{for all } \theta.$$

G. Vinnicombe
M.C. Smith

November 2025

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Version GV/3 Candidate Number:

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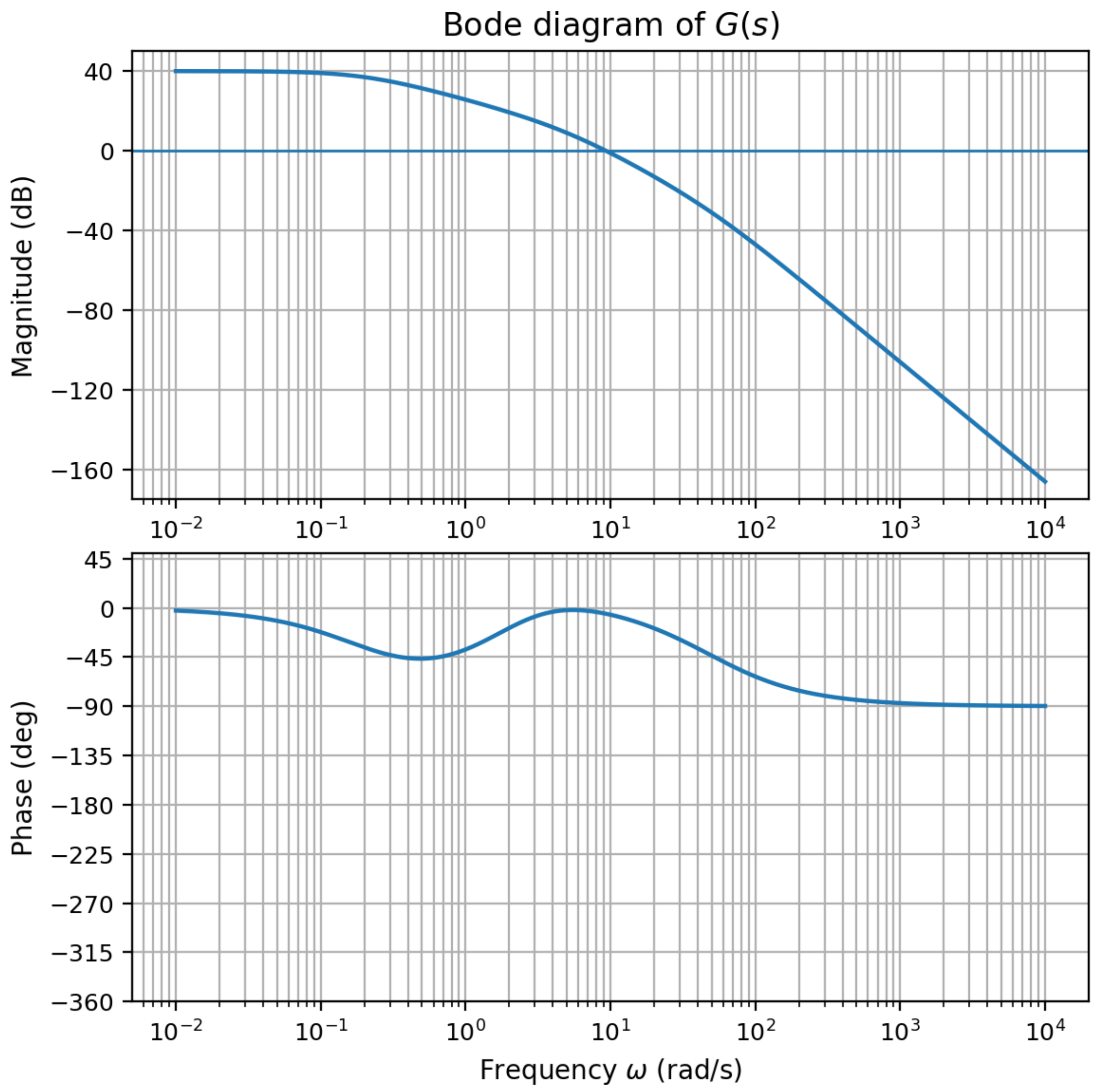
Tuesday 28 April 2026, Module 4F1, Question 3.



Extra copy of Fig. 1: Bode diagram for Question 3.

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