Q1

1. Kalman when the model is linear and Gaussian. Extended KF when model is nonlinear, differentiable, continuous state-space. Particle filter for any case, but useful mainly in nonlinear/non-Gaussian cases. If Kalman filter can be applied then particle filter should not: computationally expensive, sub-optimal (for finite N).

2. By probability chain rule and conditional probability:
$$p(x_{0:t}|y_{0:t}) = \frac{p(x_{0:t-1}|y_{0:t-1})p(x_t|x_{0:t-1}, y_{0:t-1})p(y_t|x_t, x_{0:t-1}, y_{0:t-1})}{p(y_t|y_{0:t-1})}$$
But from the Markovian assumptions of a state-space model:
$$p(x_t|x_{0:t-1}, y_{0:t-1}) = f(x_t|x_{t-1})$$
$$p(y_t|x_t, x_{0:t-1}, y_{0:t-1}) = p(y_t|x_t)$$
Hence final result.

3. The required expectation is
$$E[x_t^2|y_{0:t}] = \int x_t^2 p(x_t|y_{0:t})dx_t = \int x_t^2 p(x_{0:t}|y_{0:t})dx_{0:t} = \int x_t^2 \frac{p(x_{0:t}|y_{0:t})}{h(x_{0:t})}h(x_{0:t})dx_{0:t}$$
But, the proposed extension scheme is proposing from
$$h(x_{0:t}) = p(x_{0:t-1}|y_{0:t-1})h(x_t|x_{0:t-1})$$
, so
$$E[x_t^2|y_{0:t}] = \int x_t^2 p(x_t|y_{0:t})dx_t = \int x_t^2 p(x_{0:t}|y_{0:t})dx_{0:t} = \int x_t^2 \frac{p(x_{0:t}|y_{0:t})}{h(x_{0:t})}h(x_{0:t})dx_{0:t}$$
$$= \int x_t^2 \frac{p(x_{0:t-1}|y_{0:t-1})f(x_t|x_{t-1})g(y_t|x_t)}{p(y_t|y_{0:t-1})h(x_{0:t})}h(x_{0:t})dx_{0:t}$$
$$= \int x_t^2 \frac{f(x_t|x_{t-1})g(y_t|x_t)}{h(x_t|x_{0:t-1})p(y_t|y_{0:t-1})}\frac{1}{N}\sum_i \delta_{x_{0:t}^{(i)}} dx_{0:t}$$
$$= \frac{1}{Np(y_t|y_{0:t-1})}\sum_i x_t^{(i)2}\frac{f(x_t^{(i)}|x_{t-1}^{(i)})g(y_t|x_t^{(i)})}{h(x_t^{(i)}|x_{0:t-1}^{(i)})} = \frac{1}{Np(y_t|y_{0:t-1})}\sum_i x_t^{(i)2} w_t^{(i)}$$

4. This is estimated as
$$\frac{1}{N}\sum_i w_t^{(i)}$$
(unnormalised importance sampling)

5. Transition density:
$$f(x_t|x_{t-1}) = \mathcal{N}(\sin x_{t-1}, \sigma_u^2))$$
Observstion density:
$$y_t = |x_t|v_t$$
So by Jacobian change of variables theorem:
$$g(y_t|x_t) = 1/|x_t|\exp(-y_t/|x_t|)$$
$\hat{E}$ is as above with
$$w_t = g(y_t|x_t) = 1/|x_t|\exp(-y_t/|x_t|)$$
since the $f()$ terms cancel top and bottom.

Q2

1. Under the transition model:

$$x_t = 0.8x_{t-1} + 2v_t, \quad v_t \sim \mathcal{N}(0, 0.01)$$

we have a linear transformation of a Gaussian random variable, so that

$$\mu_{t|t-1} = 0.8\mu_{t-1|t-1} = 4.8$$

$$P_{t|t-1} = 0.8 * P_{t-1|t-1} * 0.8 + 2^2 * 0.01 = 0.16 + 0.04 = 0.2$$

and the result is normal with this mean and coviance.

2. Under the observation model

$$y_t = 2x_t + w_t, \;\; w_t \sim \mathcal{N}(0, 0.01)$$

we have a change of variables once again:

$$\mu_y = 2\mu_{t|t-1} = 9.6$$

$$P_y = 2 * P_{t|t-1} * 2 + 0.01 = 0.81$$

and again this is norrmal with this mean and covariance.

A very low value of the probability density for y_t, evaluated at the new measured datapoint, would indicate an unusual measurement, hence an outlier. This can be quantified by using e.g. the 95% confidence interval for the normal, for example.

3. From the KF recursions in steady state:

$$P = \Sigma_v + AP_{t-1|t-1}A^T = \Sigma_v + A(I - K_tB)PA^T = \Sigma_v + A(I - PB^T(BPB^T + \Sigma_w)^{-1}B)PA^T$$

$$= 0.04 + 0.8(1 - 2^2P(4P + 0.01)^{-1})0.8P$$

$$= 0.04 + 0.64P(1 - 4P(4P + 0.01)^{-1})$$

since $B = 2$, $A = 0.8$, $\Sigma_w = 2^2 * 0.01 = 0.04$ (NB scale factor of 2), $\Sigma_v = 0.01$.

Rearranging:

$$P(4P + 0.01) - 0.04(4P + 0.01) - 0.64P((4P + 0.01) - 4P) = 0$$

$$4P^2 + (0.01 - 0.16 - 0.0064)P - 0.0004 = 0$$

$$4P^2 - 0.1564P - 0.0004 = 0$$

$$P = (0.1564 \pm \sqrt{0.1564^2 + 4 * 4 * 0.0004})/8 \approx 0.1564/4 = 0.042 \approx 0.04$$

Then,

$$K = PB^T(BPB^T + \Sigma_w)^{-1} = 0.042 * 2(4 * 0.042 + 0.01)^{-1} = 0.0105$$

$$\mu_{t|t} = 0.8\mu_{t|t-1} + 0.0105 * (y_t - 2\mu_{t|t-1}) \approx 0.78\mu_{t|t-1} + 0.01y_t$$

Q3

(a)

(i) We have that $A_{ij} = 1 - \delta_{ij}$ and $D_{ij} = (n-1)\delta_{ij}$, so we get $L_{ij} = D_{ij} - A_{ij} = n\delta_{ij} - 1$.

(ii) Note $\sum_j (n\delta_{ij} - 1)v_j = nv_i - \sum_j v_j$. Let $v_i = 1/\sqrt{n}$, then the eigenvalue equation is

$$\sum_j L_{ij} v_j = \frac{n}{\sqrt{n}} - \frac{n}{\sqrt{n}} = 0 v_i$$

i.e., eigenvector with $\lambda_0 = 0$.

(iii) By orthogonality, $\sum_i v_i^{(\alpha)} v_i^{(0)} = 0$, so $\sum_i v_i^{(\alpha)}/\sqrt{n} = 0$ or $\sum_i v_i = 0$.

(iv) $\sum_j L_{ij} v_j^{(\alpha)} = nv_j^{(\alpha)} - \sum_j v_j^{(\alpha)} = nv_j^{(\alpha)}$, so eigenvalues are $\lambda^{(\alpha)} = n$.

(b)

$$\begin{aligned}
\sum_j L_{ij} v_j^{(\alpha)} &= \left(2v_i^{(\alpha)} - (v_{i+1}^{(\alpha)} + v_{i-1}^{(\alpha)})\right) \\
&= 2\cos\left(\frac{2\pi\alpha i}{n}\right) - \left(\cos\left(\frac{2\pi\alpha(i+1)}{n}\right) + \cos\left(\frac{2\pi\alpha(i-1)}{n}\right)\right) \\
&= 2\cos\left(\frac{2\pi\alpha i}{n}\right)\left(2 - 2\cos\left(\frac{2\pi\alpha}{n}\right)\right)
\end{aligned}$$

hence it is an eigenvector with eigenvalue $\lambda_\alpha = 2 - 2\cos(2\pi\alpha/n)$.

(c)

Write $y(t) = \sum_\alpha c_\alpha(t)\boldsymbol{v}^{(\alpha)}$, then

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = -\sum_\alpha c_\alpha(t)\lambda_\alpha \boldsymbol{v}^{(\alpha)}/k$$

i.e.,

$$\frac{\mathrm{d}c_\alpha}{\mathrm{d}t} = -c_\alpha(t)\lambda_\alpha/k$$

so $c_\alpha(t) = c_\alpha(0)e^{-\lambda_\alpha t/k}$.

For large t, only the terms with small λ_α contribute so

$$\boldsymbol{y}(t) \approx c_0\boldsymbol{v_0} + c_1 e^{-\lambda_1 t/k}\boldsymbol{v_1} + \dots$$

Both the pub and the restaurant converge to the same $\boldsymbol{v_0} = \text{const.}$. But, for the pub, $\lambda_1/k = n/(n-1) \approx= 1$, while for the restaurant $\lambda_1/k = 1 - \cos(2\pi/n) \approx 2\pi^2/n^2$ (large n approximation). Hence, the rate to approach the steady state is $n^2/(2\pi^2)$ larger in the pub.

Q4

(a)

$$\begin{aligned}
G(z) &= \sum_{k=0}^{\infty} z^k p_k = \frac{4}{13}\sum_{k=0}^{\infty}\left(\frac{9z}{13}\right)^k \\
&= \frac{4}{13}\frac{1}{1 - 9z/13} = \frac{4}{13 - 9z}
\end{aligned}$$

(b)

Note that:

$$G'(z) = \frac{36}{(13-9z)^2}$$

So the mean degree is $G'(1) = 36/16 = 9/4$.

The generating function for the excess degree is

$$\sum_k q_k z^k = \frac{G'(z)}{G'(1)} = \frac{16}{(13-9z)^2}$$

(c)

i) If w is the probability an edge does not lead to the giant component, then a randomly chosen edge will lead to k edges w.p. q_k, and each of these will not lead to giant component w.p. w^k, hence

$$w = \sum_{k=0}^{\infty} q_k w^k = \frac{G'(w)}{G(1)} = \frac{16}{(13-9w)^2}$$

and we see

$$w = 1/9 = \frac{16}{(13-9/9)^2}$$

so this is the solution.

ii) The probability that a node is not in the giant component is

$$\sum_{k=0}^{\infty} p_k w^k = G(1/9) = 1/3.$$

(d)

A random node with degree k has precisely one edge to the g.c. with prob $k(1-w)w^{k-1}$.

So, a random node has one connection to the g.c. with prob

$$(1-w)\sum_{k=0}^{\infty} p_k k w^{k-1} = (1-w)G'(w) = 2/9$$