EGT3
ENGINEERING TRIPOS PART IIB

Monday 2 May 2022   9.30 to 11.10

Module 4F7

STATISTICAL SIGNAL ANALYSIS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 Let

$$X_k = \alpha X_{k-1} + W_k,$$

with $X_0 = 0$, and let $W_1, W_2, \ldots$ be independent Gaussian random variables with mean zero and variance σ^2. Let

$$Y_k = X_k + V_k$$

where $V_1, V_2, \ldots$ are independent Gaussian random variables with mean zero and unit variance.

(a) Let $X = [X_1, X_2, X_3]^T$ and $Y = [Y_1, Y_2, Y_3]^T$. Find $\Sigma_x = \mathbf{E}(XX^T)$, $\Sigma_y = \mathbf{E}(YY^T)$ and $\Sigma_{yx} = \mathbf{E}(YX^T)$. [30%]

(b) An estimate of $X = [X_1, X_2, X_3]^T$ using $Y = [Y_1, Y_2, Y_3]^T$ is

$$\hat{X} = \begin{bmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \hat{X}_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

where B_1, B_2 and B_3 are the row vectors of the matrix.

(i) Let $e_i = X_i - \hat{X}_i$ for $i = 1, 2, 3$. Find the scalars b_i so that $\mathbf{E}(e_i) = 0$ and the row vectors B_i that minimise $\mathbf{E}(e_i^2)$. [30%]

(ii) Find the minimum mean square error matrix $\mathbf{E}\left((X - \hat{X})(X - \hat{X})^T\right)$. [40%]

2 Consider the model

$$Y_t = X_t + V_t$$

where $V_1, V_2, \ldots$ are independent Gaussian random variables with mean zero and variance σ^2 while $X_1, X_2, \ldots$ are independent Gaussian random variables with mean zero and variance θ^2. Assume σ^2 is known but not θ. It is desired to find the parameter θ that maximises $p_\theta(y_1, \ldots, y_T)$, the probability density function (pdf) of the observed data $y_1, \ldots, y_T$, using the Expectation-Maximisation algorithm.

(a) Find $p_\theta(x_1, y_1, \ldots, x_T, y_T)$. [20%]

(b) Find $p_\theta(x_t|y_t)$ which is the conditional pdf of X_t given $Y_t = y_t$. [20%]
[You may find the following fact useful: the Gaussian pdf with mean $-b/2a$ and variance $-1/2a$ can be expressed as $C \exp(ax^2 + bx)$.]

(c) You are given an estimate $\hat{\theta}^2$ of the variance of X_t. Evaluate the function

$$Q(\hat{\theta}, \tilde{\theta}) = \int \log\left(p_{\tilde{\theta}}(x_1) \cdots p_{\tilde{\theta}}(x_T)\right) p_{\hat{\theta}}(x_1|y_1) \cdots p_{\hat{\theta}}(x_T|y_T) dx_1 \ldots dx_T$$

where $\tilde{\theta}^2$ is an updated estimate of the unknown variance. [20%]

(d) Find $\tilde{\theta}^2$ that maximises Q. Give $\tilde{\theta}^2$ when $\sigma = 0$. [20%]

(e) Find $\tilde{\theta}^2$ that maximises $\log\left(p_{\tilde{\theta}}(x_1) \cdots p_{\tilde{\theta}}(x_T)\right)$ and verify your answer coincides with that in Part (d) when $\sigma = 0$. [20%]

3 Consider the following hidden Markov model (HMM) where X_n is a two-state hidden Markov chain, $X_n \in \{1, 2\}$, with transition probability matrix

$$P = \begin{bmatrix} 1-q & q \\ q & 1-q \end{bmatrix}.$$

Let Y_n be the observed process

$$Y_n = \begin{cases} \theta_1 W_n & \text{if } X_n = 1 \\ \theta_2 W_n & \text{if } X_n = 2 \end{cases}$$

where $W_1, W_2, \ldots$ are independent zero mean and unit variance Gaussian random variables. Assume $X_0 = x_0$.

(a) Find $p(y_n|x_n)$ for $x_n = 1$ and $x_n = 2$. [5%]

(b) Let S_n be the price of a financial asset (e.g. a share) and let $Y_n = \log(S_n/S_{n-1})$. Explain why the HMM could be suitable for this data Y_n. [5%]

(c) Let $p_n = \Pr(X_n = 1|y_1, \ldots, y_{n-1})$. Find p_{n+1}. [20%]

(d) Let $a_{n+1} = p(y_{n+1}, \ldots, y_T|X_{n+1} = 1)$ and $b_{n+1} = p(y_{n+1}, \ldots, y_T|X_{n+1} = 2)$. Find a_n and b_n. [20%]

(e) Find $\pi_n = \Pr(X_n = 1|y_1, \ldots, y_T)$. [10%]

(f) Find

$$\frac{d \log p(y_n|1)}{d\theta_i}\pi_n + \frac{d \log p(y_n|2)}{d\theta_i}(1 - \pi_n)$$

for $i = 1, 2$. [20%]

(g) Using your solution to Part (f), find the gradient of the log probability density of the data set,

$$\frac{d}{d\theta_i} \log p(y_1, \ldots, y_T)$$

for $i = 1, 2$. [20%]

4 An unknown scalar value θ is measured by a noisy sensor whose noise variance is slowly drifting over time as described by the following equations

$$
\begin{aligned}
Y_k &= \theta + X_k V_k \\
X_k &= \alpha X_{k-1} + W_k
\end{aligned}
$$

where $V_1, V_2, \ldots$ are independent Gaussian random variables with zero mean and unit variance; α is a scalar; $W_1, W_2, \ldots$ are independent zero mean Gaussian random variables with variance σ^2 and $X_0 = x_0$.

(a) Maximise $\log p(y_1|x_1)$ with respect to x_1^2 and then sketch $p(y_1|x_1)$ as x_1 varies in the interval $(-\infty, \infty)$. Sketch $p(x_1|x_0)$, for $x_0 = 0$, as x_1 varies in the interval $(-\infty, \infty)$. Conclude whether $p(x_1|y_1)$ is a unimodal or bimodal probability density function. [30%]

(b) Given $p(x_{1:t}|y_{1:t})$, find $p(x_{1:t+1}|y_{1:t+1})$ using Bayes' law. [20%]

(c) Find $d/d\theta \ \log p(x_1, y_1, \ldots, x_T, y_T)$. [20%]

(d) For $t = 1, \ldots, T$, let

$$s_t = \int \frac{1}{x_t^2} p(x_t|y_1, \ldots, y_T) dx_t.$$

Find $d/d\theta \ \log p(y_1, \ldots, y_T)$. [20%]

(e) Let $X^i_{1:T}$, $i = 1, \ldots, N$ be N independent samples from $p(x_1, \ldots, x_T|y_1, \ldots, y_T)$. Using importance sampling, find a Monte Carlo approximation of

$$\frac{d}{d\theta} \log p(y_1, \ldots, y_T, y_{T+1}).$$

[10%]

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