EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 5 May 2026 14.00 to 15.40

Module 4F7

STATISTICAL SIGNAL AND NETWORK MODELS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A general state-space model has transition density $f(x_t|x_{t-1})$ and observation density $g(y_t|x_t)$.

(a) Describe the situations where it would be appropriate to apply a Kalman filter, an extended Kalman filter, and a particle filter. In a situation where the Kalman filter can be applied, give any advantages/drawbacks from applying a particle filter. [20%]

(b) Show that the joint smoothing density may be expressed recursively as:

$$p(x_{0:t}|y_{0:t}) = \frac{p(x_{0:t-1}|y_{0:t-1})f(x_t|x_{t-1})g(y_t|x_t)}{p(y_t|y_{0:t-1})}$$

You should carefully state any Markovian assumptions that apply in your derivation. [15%]

(c) A statistician has generated random sequences $\{x_{0:t-1}^{(i)}\}$, $i = 1, ..., N$, from the distribution $p(x_{0:t-1}|y_{0:t-1})$. Each sequence $x_{0:t-1}^{(i)}$ is extended by sampling a new state $x_t^{(i)}$ from the distribution $h(x_t|x_{0:t-1}^{(i)})$. An importance sampling estimator $\hat{E} \approx E[x_t^2|y_{0:t}]$ is required. Show that this may be obtained as

$$\frac{1}{Np(y_t|y_{0:t-1})}\sum_{i=1}^{N}\left(x_t^{(i)}\right)^2 \frac{f(x_t^{(i)}|x_{t-1}^{(i)})g(y_t|x_t^{(i)})}{h(x_t^{(i)}|x_{0:t-1}^{(i)})}$$

[25%]

(d) For the model chosen it is difficult to calculate $p(y_t|y_{0:t-1})$. How may this be approximated within the Importance Sampling scheme? [10%]

(e) The transition model used was:

$$x_t = \sin(x_{t-1}) + u_t, \;\; u_t \sim \mathcal{N}(0, \sigma_u^2)$$

and the observation equation was

$$y_t = |x_t|\, v_t$$

where $p(v_t) = \exp(-v_t)$ for $v_t > 0$. The proposal function $h(.|.)$ was set equal to the transition density $f(.|.)$.

Write down the transition density, the observation density, and the importance sampling estimator $\hat{E}$ in this specific case. [30%]

2 Suppose that for some particular time $t-1$, $p(x_{t-1}|y_{0:t-1}) = \mathcal{N}(x_{t-1}|6, 0.25)$.

The next state is generated from the model:

$$x_t = 0.8x_{t-1} + 2v_t, \quad v_t \sim \mathcal{N}(0, 0.01)$$

where v_t is drawn independently.

The new data point is generated as

$$y_t = 2x_t + w_t, \quad w_t \sim \mathcal{N}(0, 0.01)$$

where w_t is also drawn independently.

(a) Show from the state equations above and probability theory that the predictive distribution is

$$p(x_t|y_{0:t-1}) = \mathcal{N}(x_t|4.8, 0.2)$$

[30%]

(b) Show from state equations above and probability theory that the incremental likelihood is:

$$p(y_t|y_{0:t-1}) = \mathcal{N}(y_t|9.6, 0.81)$$

Describe how this term could be used to detect 'outliers', i.e. data points that do not fit the stated model. [30%]

(c) Under the stated model at the start of the question, it is suggested that the predictive covariance $P_{t|t-1}$ might settle to a steady-state value P at some time t in the distant future. Show that the steady-state value P is approximately 0.04. At this steady-state, show that the Kalman filter recursion becomes

$$\mu_{t|t} = \alpha y_t + \beta \mu_{t-1|t-1}$$

where α and β should be determined. [40%]

Recall that the Kalman filtering recursions for a linear state-space model, based on data $y_0, y_1, y_2, \ldots y_t$, *are expressed as*

$$\begin{aligned}
\mu_{t|t-1} &= A\mu_{t-1|t-1} \\
P_{t|t-1} &= \Sigma_v + AP_{t-1|t-1}A^T \\
\mu_{t|t} &= \mu_{t|t-1} + K_t(y_t - B\mu_{t|t-1}) \\
P_{t|t} &= (I - K_tB)P_{t|t-1} \\
K_t &= P_{t|t-1}B^T(BP_{t|t-1}B^T + \Sigma_w)^{-1}
\end{aligned}$$

where the terms $\mu_{t|t-1}$ *and* $\mu_{t|t}$ *are estimators of a hidden state variable* x_t.

3 A group of n students are planning a social event. They are deciding between meeting in the pub or meeting for dinner at a restaurant.

If the students meet at the pub, all n students will spend a proportion of their time talking to each of the $n-1$ other students. Hence, their social graph would be the complete graph on n nodes.

Conversely, if the students meet at the restaurant, they will sit around a large table and will only talk to the people immediately to their right and left. Hence, their social graph would be a cycle on n nodes.

Let $L = D - A$ denote the graph Laplacian. The orthogonal eigenvectors $\boldsymbol{v}^{(\alpha)}$ are defined by $L\boldsymbol{v}^{(\alpha)} = \lambda_\alpha \boldsymbol{v}^{(\alpha)}$ for $\alpha = 0, 1, \ldots, n-1$.

(a) Assume the students meet at the pub.

(i) Show that the Laplacian is $L_{ij} = n\,\delta_{ij} - 1$.

(ii) Show that $v_i^{(0)} = n^{-1/2}$ for all i is an eigenvector with eigenvalue $\lambda_0 = 0$.

(iii) Using orthogonality, show that for $\alpha \geq 1$, $\sum_i v_i^{(\alpha)} = 0$.

(iv) Hence, argue that $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = n$.

[30%]

(b) Assume instead that the students meet at the restaurant. Show that

$$v_i^{(\alpha)} = \cos\left(\frac{2\pi\alpha i}{n}\right)$$

are eigenvectors of L with associated eigenvalues

$$\lambda_\alpha = 2 - 2\cos\left(\frac{2\pi\alpha}{n}\right)$$

[30%]

(c) When the students first meet, at $t = 0$, gossip is distributed so that person i has $y_i(0)$ information. Assume that gossip diffuses through the network so that

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = -L\boldsymbol{y}/k$$

where k is the mean degree. Show that the solution is $\boldsymbol{y}(t) = \sum_{\alpha=0}^{n-1} e^{-\lambda_\alpha t/k} c_\alpha \boldsymbol{v}^{(\alpha)}$ and state an expression for the constants c_α. By looking at the behaviour of $\boldsymbol{y}(t)$ for large times argue that gossip spreads approximately $n^2/2\pi^2$ times faster at the pub than the restaurant.

[40%]

4 In this question we consider random graphs with degree distribution

$$p_k = \left(\frac{4}{13}\right)\left(\frac{9}{13}\right)^k$$

(a) Show that the generating function for the degree distribution is

$$G(z) = \frac{4}{13 - 9z}$$

[20%]

(b) Hence or otherwise, show that the mean degree is $9/4$, and that the generating function for the excess degree distribution is

$$\frac{16}{(13 - 9z)^2}$$

[20%]

(c) Let w be the probability that an edge chosen uniformly at random does not lead to the giant component.

(i) Show that $w = 1/9$.

(ii) What is the fractional size of the giant component?

[30%]

(d) What fraction of nodes have precisely one edge that leads to the giant component? [30%]

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