(a) The Jacobian $J \neq 0$, the signs of the expressions $B^2 - AC$ and $b^2 - ac$ are the same
$\Rightarrow$ The type of a second order PDE dose not change under coordinate transformations

$\Delta > 0$ hyperbolic, $\Delta = 0$ parabolic, $\Delta < 0$ elliptic.

10%

(b) $a = 1$, $b = -\cos x$, $c = 2 - \sin^2 x$

$\Delta = b^2 - ac = \cos^2 x - 2 + \sin^2 x = -1 < 0$

$\Rightarrow$ Elliptic Eqn.

$\frac{dy}{dx} = -\cos x + i \Rightarrow y = -\sin x + ix + \xi - i\eta$

Hence $\boxed{\xi = \sin x + y, \quad \eta = x}$

Derivatives: $\frac{\partial \xi}{\partial x} = \cos x$, $\frac{\partial \xi}{\partial \eta} = 1$, $\frac{\partial \eta}{\partial x} = 1$, $\frac{\partial \eta}{\partial y} = 0$

chain rule: $u_x = \cos x \, u_\xi + u_\eta$

$u_y = 1 \times u_\xi + 0 \times u_\eta = u_\xi$

$u_{xx} = (\cos x \, u_\xi)_x + (u_\eta)_x$

$= -\sin x \, u_\xi + \cos x \, (u_\xi)_x + u_{\eta\xi} \cos x + u_{\eta\eta}$

$= -\sin x \, u_\xi + \cos^2 x \, u_{\xi\xi} + 2\cos x \, u_{\xi\eta} + u_{\eta\eta}$

$u_{xy} = (u_y)_x = (u_\xi)_x = \cos x \, u_{\xi\xi} + u_{\xi\eta}$

$u_{yy} = (u_\xi)_y = u_{\xi\xi}$

Substitute into the original eqn:

$-\sin x \, u_\xi + \cos^2 x \, u_{\xi\xi} + \cancel{2\cos x \, u_{\xi\eta}} + u_{\eta\eta} - 2\cos^2 x \, u_{\xi\xi}$

$\cancel{-2\cos x \, u_{\xi\eta}} + (2 - \sin^2 x) \, u_{\xi\xi} + u = 0$

$\boxed{u_{\xi\xi} + u_{\eta\eta} + u - u_\xi \sin\eta = 0}$

40%

$b^2 - ac < 0$, elliptic type

Q1

(c) The coefficients must satisfy

20%

$$A = C, \quad B = 0$$

(d) The integration constant $z = \xi + i\eta$ of the characteristic equation must satisfy

$$a(z_x)^2 + 2b\, z_x z_y + c\,(z_y)^2 = 0$$

or

$$a(\xi_x^2 - \eta_x^2) + 2b(\xi_x \xi_y - \eta_x \eta_y) + c(\xi_y^2 - \eta_y^2)$$

$$+ 2i\,[a\,\xi_x \eta_x + b(\xi_x \eta_y + \eta_x \xi_y) + c\,\xi_y \eta_y] = 0$$

Hence $a\,\xi_x^2 + 2b\,\xi_x \xi_y + c\,\xi_y^2 = a\,\eta_x^2 + 2b\,\eta_x \eta_y + c\,\eta_y^2$

$$a\,\xi_x \eta_x + b(\xi_x \eta_y + \xi_y \eta_x) + c\,\xi_y \eta_y = 0$$

From Eqn. (1), we obtain

$$\boxed{A = C, \quad B = 0}$$

30%

The Equation is $\boxed{\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial y^2}}$ Q2

(a) The problem has ~~neither~~ neither time scale nor length scale, hence may have a similarity solution. The temperature profile at different times $T(t,*)$ are different, but if we rescale the ~~y~~ coordinate by an appropriate length $\ell = \sqrt{\kappa t}$, the profile will be the same, this can be expressed in form

$$\frac{T - T_0}{T_1 - T_0} = f(\eta), \quad \eta = y/\sqrt{\kappa t}.$$ 20%

(b) In spherical coordinate systeme

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial x}\right) + \cdots$$

By symmetry, the equation is

$$\frac{\partial T}{\partial t} = \kappa \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)$$ 10%

(c) $y = r - R$, $\frac{dy}{dr} = 1$.

$\Gamma = r(T - T_0)$, $T = T_0 + \frac{\Gamma}{r}$, $\frac{\partial T}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\Gamma}{r}\right) = \frac{1}{r}\frac{\partial \Gamma}{\partial r} - \frac{\Gamma}{r^2}$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Gamma}{\partial r}\right) = \frac{\partial}{\partial r}\left(r\frac{\partial \Gamma}{\partial r} - \Gamma\right) = r\frac{\partial^2 \Gamma}{\partial r^2} + \cancel{\frac{\partial \Gamma}{\partial r}} - \cancel{\frac{\partial \Gamma}{\partial r}}$$

Hence

$$\frac{1}{r}\frac{\partial \Gamma}{\partial t} = \kappa \frac{1}{r^2} r \frac{\partial^2 \Gamma}{\partial y^2}$$

we obtain $\boxed{\frac{\partial \Gamma}{\partial t} = \kappa \frac{\partial^2 \Gamma}{\partial r^2}}$ 20%

(d) The equation in (c) is the same as in (a)
Hence we can look for similarity solution

$$\Gamma = f(\eta) \quad , \quad \eta = y/\sqrt{\kappa t} \qquad \frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{\eta}{t}$$

$$\frac{\partial f}{\partial t} = \frac{df}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{\eta}{2t}\frac{df}{d\eta}$$

$$\frac{\partial f}{\partial y^2} = \frac{d^2 f}{d\eta^2}\frac{1}{\kappa t}$$

Hence $$-\frac{\eta}{2t}\frac{df}{d\eta} = \kappa \frac{1}{\kappa t}\frac{d^2 f}{d\eta^2}$$

$$\Rightarrow \boxed{\frac{d^2 f}{d\eta^2} + \frac{1}{2}\eta \frac{df}{d\eta} = 0}$$

30%

(e) set $(\ln f')' = f''/f' = -\frac{\eta}{2}$

Integrate $\ln(f') = -\frac{\eta^2}{4} + A$

or $f' = A e^{-\frac{\eta^2}{4}}$

Finally $f(\eta) = A\int_0^\eta e^{-\eta^2/4}\, d\eta + B$

For $\eta = 0$, $\Gamma(0) = f(0) = B = \Gamma(0) = R(T_i - T_0)$

For $\eta = \infty$, $\Gamma(\infty) = 0 = A\int_0^{+\infty} e^{-\eta^2/4}\, d\eta + B$

$$\Rightarrow A = -\frac{B}{\sqrt{\pi}}$$

Hence $\Gamma = f(\eta) = R(T_i - T_0)\left[1 - \frac{1}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2/4}\, d\eta\right]$

$$\frac{T - T_0}{T_i - T_0} = \frac{R}{r}\left[1 - \frac{1}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2/4}\, d\eta\right]$$

20%

3 The current state of a battery is described by its level of charge, Q, and its temperature T. At time $t = 0$ the battery starts uncharged, $Q = 0$, and is at room temperature $T = T_R$. We then charge the battery by applying a controlled current $I(t) = \frac{dQ}{dt}$, which leads to a change in temperature

$$\frac{dT}{dt} = \alpha I(t)^2 + \beta(T_R - T).$$

After a time t_F, the battery is required to be fully charged, $Q = Q_F$. We wish to find the charging profile $I(t)$ that achieves this with minimum final temperature $T(t_F)$.

(a) Discuss briefly the likely physical origins of the two terms that make up $\frac{dT}{dt}$? [10%]

The αI^2 term is resistive heating ($P = I^2 R$) associated with the internal resistance of the battery. The second term describes cooling (heat dissipation) driven by the temperature difference between the battery and its environment (Newton's law of cooling).

(b) Show that minimizing $T(t_F)$ is equivalent to minimizing an augmented functional of the form

$$J = \int_0^{t_F} (1 - \lambda_1)\frac{dT}{dt} + \lambda_1(\alpha I^2 + \beta(T_R - T)) + \lambda_2 I \mathrm{d}t - \lambda_2 Q_F,$$

[20%]

and comment on the origin and character of λ_1 and λ_2.

This is a control problem. The object to minimize is choose $I(t)$ such that we minimize $T(t_F) = T_R + \int_0^{t_F} \frac{dT}{dt}\mathrm{d}t$.

This minimization is subject to two constraints:

Firstly, the total charge supplied must be Q_F, i.e. $\int_0^{t_F} I\mathrm{d}t = Q_F$.

Secondly, at each point in time, we must obey the heating/cooling equation, $\alpha I(t)^2 + \beta(T_R - T) - \frac{dT}{dt} = 0$.

We use the method of Lagrange multipliers, and formulate the augmented functional $J = T(t_F) + \int_0^{t_F} \lambda_1(t)\left(\alpha I(t)^2 + \beta(T_R - T) - \frac{dT}{dt}\right)\mathrm{d}t + \lambda_2\left(\int_0^{t_F} I(t)\mathrm{d}t - Q_F\right)$.

We note that λ_1 and λ_2 are both Lagrange multipliers, but λ_1 is a function of time, as the heating/cooling equation must be satisfied at each point in time, whereas λ_2 is a single value as the total charge being Q_F need only be true at the end of the charging process.

Combining the integrals and disregarding the irrelevant constant T_R puts J into the form given in the question.

(c) Find the full set equations and boundary conditions that must be satisfied by T, I, λ_1 and λ_2 to minimize $T(t_F)$. [30%]

We need to minimize with respect to $I(t)$, $T(t)$, $\lambda_1(t)$ and λ_2. Defining the integrand $F(I, T, \dot{T}, \lambda_1) = (1-\lambda_1)\dot{T} + \lambda_1(\alpha I^2 + \beta(T_R - T)) + \lambda_2 I$, such that $J = \int_0^{t_F} F\mathrm{d}t - \lambda_2 Q_F$, we consider the E-L equation with respect to each functional quantity:

$$I: \quad \frac{\partial F}{\partial I} = 0 \rightarrow 2\lambda_1 \alpha I + \lambda_2 = 0 \tag{2}$$

$$T: \quad \frac{d}{dt}\frac{\partial F}{\partial \dot{T}} - \frac{\partial F}{\partial T} = 0 \rightarrow -\dot{\lambda}_1 + \beta\lambda_1 = 0 \tag{3}$$

$$\lambda_1: \quad \frac{\partial F}{\partial \lambda_1} = 0 \rightarrow -\dot{T} + \alpha I^2 + \beta(T_R - T) = 0. \tag{4}$$

As λ_2 is a constant, in this case, we just take the derivative of J directly to get the relevant constraint equation:

$$\lambda_2: \quad \frac{dJ}{d\lambda_2} = 0 \rightarrow \int_0^{t_F} I(t)\mathrm{d}t = Q_F.$$

Only the $E-L$ equation for T arises via integration by parts, yielding boundary conditions. At $t = 0$ we have the fixed starting condition $T(0) = T_R$. At t_F there is no constraint, so we have the natural boundary condition $\frac{\partial F}{\partial \dot{T}} = 0 \rightarrow \lambda_1(t_F) = 1$.

(d) Show that the optimal charging profile has the form $I(t) = A\exp(-\beta t)$, and the corresponding temperature profile has the form $T(t) = T_R + B(\exp(-\beta t) - \exp(-2\beta t))$, and explain how you would also find the values of the constants A and B. You do not need to actually find the value of A and B. [30%]

Solving eqn. 3 above gives $\lambda_1 = C\exp(\beta t)$, with C being a constant of integration. Inserting this into eqn. 2, we obtain $I = -\lambda_2/(2\alpha\lambda_1) \equiv A\exp(-\beta t)$. To find the value of A, we would now substitute into the Q_F constraint.

Substituting this result for I into eqn. 4 we get an equation for T:

$$\dot{T} + \beta T = \beta T_R + \alpha A^2 \exp(-2\beta t).$$

The solution requires a particular integral and a complementary function. The complementary function solves $\dot{T} + \beta T = 0$, giving $T_{CF} = C\exp(-\beta t)$, C being another constant of integration. The particular integral is of the form $T_{PI} = T_R - B\exp(-2\beta t)$, where B could be found by substitution, but is not required. The full solution therefore has the form

$$T = T_R + C\exp(-\beta t) - B\exp(-2\beta t).$$

Finally, requiring $T(0) = T_R$ fixes this to the constant of integration $C = B$, so we have $T = T_R + B(\exp -\beta t - \exp(-2\beta t))$ as stated, and the value of B would follow from substituting the form into 4.

(e) In reality it is the maximum temperature during charging that matters, which may or may-not occur at the end. For this class of solutions, find the condition on t_F that dictates whether the maximum temperature is during or at the end of the charging process. [10%]

The temperature profile is $T = T_R + B(\exp(-\beta t) - \exp(-2\beta t))$. *If the maximum temperature is during charging there will be a stationary point. Differentiating to find a maximum thus requires* $\dot{T} \propto -\beta \exp -\beta t + 2\beta \exp(-2\beta t) = 0$*, which is solved at* $\beta t = \log 2$*. This will occur before the end of charging unless* $t_F \leq \log 2/\beta$*.*

4 We have an experimentally obtained measurement of a scalar field $u_0(x, y, z)$ over the finite 3D domain Ω, with boundary surface $\partial\Omega$. The experimental measurement contains noise, so we try to find a smoother representation, $u(x, y, z)$, by minimizing the functional

$$J = \int_\Omega \left((u - u_0)^2 + \lambda \nabla u \cdot \nabla u\right) \mathrm{d}V,$$

where λ is a fixed positive value, not subject to minimization.

(a) Rewrite J using index notation and the summation convention, and discuss briefly the likely effect on u of varying the value of λ between small and large values? [15%]

Using index notation, with $x_i = x, y, z$ for $i = 1, 2, 3$,

$$J = \int_\Omega \left((u - u_0)^2 + \lambda \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}\right) \mathrm{d}V.$$

The first term in the integrand promotes fidelity of u to u_0, and is minimized at $u = u_0$. The second term penalizes gradients in u, and is minimized by constant (flat) u. The parameter λ changes the balance between these two with $\lambda = 0$ giving a trivial minimum at u_0 while $\lambda \to \infty$ constrains u to be flat, and hence the minimum to be uniform: $u = c$.

(b) By taking the directional derivative of J with respect to u, find both the differential equation and boundary conditions that must be satisfied for u to minimize J. [30%]

Taking the directional derivative of J with respect to u in the direction of the variation w requires us to evaluate:

$$\left.\frac{d(J(u + \epsilon w))}{d\epsilon}\right|_{\epsilon=0} = \int_\Omega \frac{d}{d\epsilon}\left.\left((u + \epsilon w - u_0)^2 + \lambda \frac{\partial(u + \epsilon w)}{\partial x_i}\frac{\partial(u + \epsilon w)}{\partial x_i}\right)\right|_{\epsilon=0} \mathrm{d}V \tag{5}$$

$$= \int_\Omega \left(2(u - u_0)w + 2\lambda \frac{\partial w}{\partial x_i}\frac{\partial u}{\partial x_i}\right) \mathrm{d}V \tag{6}$$

We integrate the second term by parts, using the divergence theorem

$$\left.\frac{d(J(u + \epsilon w))}{d\epsilon}\right|_{\epsilon=0} = \int_\Omega \left(2(u - u_0) - 2\lambda \frac{\partial^2 u}{\partial x_i \partial x_i}\right) w \mathrm{d}V + \int_{\partial\Omega} w \frac{\partial u}{\partial x_i} n_i \mathrm{d}A$$

where n_i is the outward unit normal on the boundary. This derivative must vanish for every function w over Ω. Hence the integrand must vanish, giving us the bulk pde:

$$u - u_0 = \lambda \frac{\partial^2 u}{\partial x_i \partial x_i} \equiv \lambda \nabla^2 u.$$

Since there are no constraints on u at the boundary, w may take any value on the boundary, so for the boundary part of the derivative to vanish we must have the natural boundary condition $\frac{\partial u}{\partial x_i} n_i = 0$, or, in vector notation, $\nabla u \cdot \mathbf{n} = 0$.

(c) Find the weak form of the minimization problem, and hence show that u and u_0 will have the same average over the domain, that is $\int_\Omega u \mathrm{d}V = \int_\Omega u_0 \mathrm{d}V$. [30%]

To find the weak form, we multiply the pde above by a weight function w and integrate over the domain, giving:

$$\int_\Omega \left((u - u_0) - \lambda \frac{\partial^2 u}{\partial x_i \partial x_i} \right) w \mathrm{d}V = 0$$

We next integrate the second term by parts using the divergence theorem:

$$\int_\Omega \left((u - u_0) w + \lambda \frac{\partial w}{\partial x_i} \frac{\partial u}{\partial x_i} \right) \mathrm{d}V - \int_{\partial\Omega} w \lambda \frac{\partial u}{\partial x_i} n_i \mathrm{d}A = 0.$$

Inserting our boundary condition, the surface integral vanishes, so we just have

$$\int_\Omega \left((u - u_0) w + \lambda \frac{\partial w}{\partial x_i} \frac{\partial u}{\partial x_i} \right) \mathrm{d}V = 0.$$

This is the weak form. In hindsight, this it is exactly the same as the first line in our calculation of the directional derivative of J, as expected since the problem is actually a variational one.

The weak form is true for all weight functions w. If we consider the special case $w = 1$, the derivative $\frac{\partial w}{\partial x_i} = 0$, so we have

$$\int_\Omega ((u - u_0)) \, \mathrm{d}V = 0 \rightarrow \int_\Omega u \mathrm{d}V = \int_\Omega u_0 \mathrm{d}V$$

as required by the question.

(d) An approximate solution is given by setting $u = c$ to a uniform value. Use the Rayleigh-Ritz method to find the optimal value of c. [10%]

Inserting $u = c$ into J, we have

$$J = \int_\Omega (c - u_0)^2 \, \mathrm{d}V$$

Minimizing with respect to c requires

$$\frac{dJ}{dc} = 0 \rightarrow \int_\Omega 2 (c - u_0) \, \mathrm{d}V = 0 \rightarrow c = \frac{1}{V} \int_\Omega u_0 \mathrm{d}V,$$

where V is the volume of the domain.

(e) It is proposed to replace the $\nabla u \cdot \nabla u$ in J by $\left(\nabla^2 u\right)^2$. Derive the new partial differential equation for u? (You do not need to work out the corresponding boundary conditions). [15%]

We now have:

$$J = \int_\Omega \left((u - u_0)^2 + \lambda \frac{\partial^2 u}{\partial x_i x_i} \frac{\partial^2 u}{\partial x_j \partial x_j} \right) \mathrm{d}V.$$

We compute the same directional derivative as earlier:

$$\left.\frac{d(J(u + \epsilon w))}{d\epsilon}\right|_{\epsilon=0} = \int_\Omega \frac{d}{d\epsilon} \left((u + \epsilon w - u_0)^2 + \lambda \frac{\partial^2 (u + \epsilon w)}{\partial x_i x_i} \frac{\partial^2 (u + \epsilon w)}{\partial x_j x_j} \right)\Bigg|_{\epsilon=0} \mathrm{d}V \quad (7)$$

$$= \int_\Omega \left(2(u - u_0)w + 2\lambda \frac{\partial^2 w}{\partial x_i x_i} \frac{\partial^2 u}{\partial x_j x_j} \right) \mathrm{d}V \quad (8)$$

We must now integrate the second term by parts twice, using the divergence theorem. This produces two sign flips, leaving a + sign overall. We are invited to not write out the boundary terms, so we can jump straight to

$$\left.\frac{d(J(u + \epsilon w))}{d\epsilon}\right|_{\epsilon=0} = \int_\Omega \left(2(u - u_0)w + 2\lambda w \frac{\partial^2}{\partial x_i x_i} \frac{\partial^2 u}{\partial x_j x_j} \right) \mathrm{d}V + \text{boundary terms.}$$

Factorizing out w and requiring the integrand to vanish for all functions w, we obtain a biharmonic pde:

$$u - u_0 = -\lambda \frac{\partial^2}{\partial x_i x_i} \frac{\partial^2 u}{\partial x_j x_j} = -\lambda \nabla^2 (\nabla^2 u).$$