Version JSB/4

EGT3/EGT2
ENGINEERING TRIPOS PART IIB
ENGINEERING TRIPOS PART IIA

Friday 1st May 2026   14.00 to 15.40

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A second-order linear PDE is defined as

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} = f\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, u, x, y\right)$$

If a change of variables $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ is made, the PDE reads

$$A\frac{\partial^2 u}{\partial \xi^2} + 2B\frac{\partial^2 u}{\partial \xi \partial \eta} + C\frac{\partial^2 u}{\partial \eta^2} = G\left(\xi, \eta, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, u\right)$$

where

$$\begin{aligned} A(\xi,\eta) &= a\left(\frac{\partial \xi}{\partial x}\right)^2 + 2b\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + c\left(\frac{\partial \xi}{\partial y}\right)^2 \\ B(\xi,\eta) &= a\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + b\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + c\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} \\ C(\xi,\eta) &= a\left(\frac{\partial \eta}{\partial x}\right)^2 + 2b\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + c\left(\frac{\partial \eta}{\partial y}\right)^2 \end{aligned} \tag{1}$$

(a) It can be shown that

$$B^2 - AC = (b^2 - ac)J^2$$

where J is the Jacobian of the coordinate transformation. What can be deduced from the above equation? [10%]

(b) Classify the following PDE and reduce it to its standard form.

$$\frac{\partial^2 u}{\partial x^2} - 2\cos x\frac{\partial^2 u}{\partial x \partial y} + \left(2 - \sin^2 x\right)\frac{\partial^2 u}{\partial y^2} + u = 0$$

[40%]

(c) For the case where $b^2 - ac < 0$, what conditions must the coefficients A, B and C in Eqn. (1) satisfy so that the new coordinates ξ and η reduce the general equation to its standard form? [20%]

(d) For the case introduced in (c), the characteristic equations

$$\frac{dy}{dx} = \frac{b \pm i\sqrt{ac - b^2}}{a}$$

are complex. Integrating these equations yields complex constants of integration, which we define as $\xi \pm i\eta$. Show that using ξ and η as our new coordinates reduces the general equation to its standard form. [30%]

2 The diffusion equation is given as

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

where T is the temperature and κ the diffusivity coefficient.

(a) Consider a one dimensional time-dependent temperature distribution $T(t, y)$ in the semi-infinitely long metal rod $y \geq 0$: $T(0, y) = T_0$, for $y > 0$; $T(t, 0) = T_1$, for $t \geq 0$; T_0 and T_1 are two constants. Explain why a similarity solution may exist and write down a mathematical formula to express the similarity condition. [20%]

(b) A spherical gas cavity of radius R is embedded in an infinite, thermally conducting solid whose initial temperature is T_0. At time $t = 0$, the gas temperature is raised from T_0 to T_1 and fixed at T_1. Write down the spherically symmetric diffusion equation for the temperature $T(t, r)$, where r is the radial distance from the centre of the cavity. [10%]

(c) To solve the equation in part (b), it is convenient to first make the change of variables $y = r - R$ and $\Gamma(t, y) = r\left[T(t, r) - T_0\right]$. Deduce the equation for Γ. [20%]

(d) By considering similarity solutions of the form

$$\Gamma = f(\eta)\,, \quad \eta = y/(\kappa t)^{1/2}$$

convert the partial differential equation for Γ found in part (c) into an ordinary differential equation for f. [30%]

(e) Use the differential equation from part (d) to find the temperature distribution, $T(t, r)$ that solves the original problem introduced in part (b). Your answer should be expressed in terms of the integral $\int \exp\left(-\eta^2/4\right) d\eta$. [20%]

You may use the data book and the fact that $\int_0^\infty \exp\left(-\eta^2/4\right) d\eta = \sqrt{\pi}$.

3 The present state of a battery is described by its level of charge, Q, and its temperature T. At time $t = 0$ the battery starts uncharged, $Q = 0$, and is at room temperature $T = T_R$. We then charge the battery by applying a controlled current $I(t) = \frac{dQ}{dt}$, which leads to a change in temperature

$$\frac{dT}{dt} = \alpha I(t)^2 + \beta(T_R - T).$$

After a time t_F, the battery is required to be fully charged, $Q = Q_F$. We wish to find the charging profile $I(t)$ that achieves this with minimum final temperature $T(t_F)$.

(a) Discuss briefly the likely physical origins of the two terms that make up $\frac{dT}{dt}$? [10%]

(b) Show that minimizing $T(t_F)$ is equivalent to minimizing an augmented functional of the form

$$J = \int_0^{t_F} \left[(1 - \lambda_1)\frac{dT}{dt} + \lambda_1(\alpha I^2 + \beta(T_R - T)) + \lambda_2 I \right] \mathrm{d}t - \lambda_2 Q_F,$$

and comment on the origin and character of λ_1 and λ_2. [20%]

(c) Find the full set of equations and boundary conditions that must be satisfied by T, I, λ_1 and λ_2 to minimize $T(t_F)$. [30%]

(d) Show that the optimal charging profile has the form $I(t) = A\exp(-\beta t)$, and the corresponding temperature profile has the form $T(t) = T_R + B\left[\exp(-\beta t) - \exp(-2\beta t)\right]$, and explain how you would also find the values of the constants A and B. You do not need to actually find the value of A and B. [30%]

(e) In reality it is the maximum temperature during charging that matters, which may or may not occur at the end. For the class of solutions described in part (d) find the condition on t_F that dictates whether the maximum temperature is during or at the end of the charging process. [10%]

4 We have an experimentally obtained measurement of a scalar field $u_0(x, y, z)$ over the finite 3D domain Ω, with boundary surface $\partial\Omega$. The experimental measurement contains noise, so we try to find a smoother representation, $u(x, y, z)$, by minimizing the functional

$$J = \int_{\Omega} \left((u - u_0)^2 + \lambda \nabla u \cdot \nabla u \right) \mathrm{d}V,$$

where λ is a fixed positive value, not subject to minimization.

(a) Rewrite J using index notation and the summation convention, and discuss briefly the likely effect on u of varying the value of λ between small and large values? [15%]

(b) By taking the directional derivative of J with respect to u, find both the differential equation and boundary conditions that must be satisfied for u to minimize J. [30%]

(c) Find the weak form of the minimization problem, and hence show that u and u_0 will have the same average over the domain, that is $\int_{\Omega} u \mathrm{d}V = \int_{\Omega} u_0 \mathrm{d}V$. [30%]

(d) An approximate solution is given by setting $u(x, y, z) = c$, where c is a constant. Use the Rayleigh-Ritz method to find the optimal value of c. [10%]

(e) It is proposed to replace the $\nabla u \cdot \nabla u$ in J by $\left(\nabla^2 u\right)^2$. Derive the new partial differential equation for u. (You do not need to work out the corresponding boundary conditions). [15%]

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