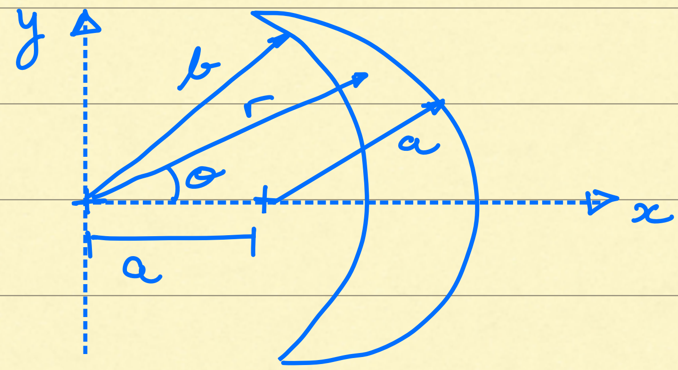
3C7/2026

1.



(a) Inner surface $r = b$

$$\psi = c\left(1 - \frac{b^2}{b^2}\right)\left(b^2 - 2ab\cos\theta\right) = 0 \checkmark$$

Outer surface $r = 2a\cos\theta$

$$\psi = \left(1 - \frac{b^2}{4a^2\cos^2\theta}\right)\left(4a^2\cos^2\theta - 4a^2\cos^2\theta\right) = 0 \checkmark$$

$$\nabla^2\psi = \left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)$$

$$\frac{\partial\psi}{\partial r} = c\left(2r - 2a\cos\theta - \frac{2ab^2}{r^2}\cos\theta\right)$$

$$\frac{\partial^2\psi}{\partial r^2} = c\left(2 + \frac{4ab^2}{r^3}\cos\theta\right)$$

$$\frac{\partial^2\psi}{\partial\theta^2} = 2c\left(1 - \frac{b^2}{r^2}\right)ar\cos\theta$$

$$\Rightarrow \nabla^2 \psi = c\left[2 + \cancel{\frac{4ab^2}{r^3}\cos\theta} + \right.$$

$$2 - \cancel{\frac{2a}{r}\cos\theta} - \cancel{\frac{2ab^2}{r^3}\cos\theta} +$$

$$\left. \cancel{\frac{2a}{r}\cos\theta} - \cancel{\frac{2ab^2}{r^3}\cos\theta} \right)$$

$$= 4C = -2G\alpha$$

$$\text{ie} \quad C = -\frac{G\alpha}{2}$$

$\psi = 0$ on boundary & $\nabla^2\psi =$ constant $\Rightarrow$
$\psi =$ Prandtl stress function.

(b) In cartesian co-ordinates

$$\psi = -\frac{G\alpha}{2}\left[1 - \frac{b^2}{x^2+y^2}\right]\left[x^2+y^2 - 2ax\right]$$

$$= -\frac{G\alpha}{2}\left[x^2+y^2 + \frac{2ab^2x}{x^2+y^2} - 2ax - b^2\right]$$

$$\sigma_{zy} = -\frac{\partial \psi}{\partial x}$$

$$\sigma_{zy} = G\alpha\left[x - a + \frac{ab^2}{x^2+y^2} - \frac{2ab^2x^2}{(x^2+y^2)^2}\right]$$

(c) $$\sigma_{zy}(b,0) = G\alpha\left[b - a + a - 2a\right]$$

$$= -G\alpha a\left(2 - \frac{b}{a}\right)$$

$$\sigma_{zy}(2a,0) = G\alpha\left[2a - a + \frac{b^2}{4a} - \frac{b^2}{2a}\right]$$

$$= G\alpha a\left[1 - \frac{b^2}{4a^2}\right]$$

(d) Let $\frac{b}{a} = 2 - \beta$

$$\Rightarrow \sigma_{zy}(b,0) = -G\alpha a\,\beta$$

$$\sigma_{zy}(2a,0) = G\alpha a\left(1 - \frac{b}{2a}\right)\left(1 + \frac{b}{2a}\right)$$

$$= G\alpha a\left(1 - \frac{2-\beta}{2}\right)\left(1 + \frac{2-\beta}{2}\right)$$

$$= G\alpha a\left(\beta - \frac{\beta^2}{4}\right)$$

$$\Rightarrow \left|\sigma_{zy}(b,0)\right| > \left|\sigma_{zy}(2a,0)\right|$$

2.

(a) Thin disk $\Rightarrow$ plane stress.

For annulus Lame equations

$$\sigma_r = A - \frac{B}{r^2}, \qquad \sigma_\theta = A + \frac{B}{r^2}$$

$$\sigma_r(a) = -p = A - \frac{B}{a^2}$$

$$\sigma_r(b) = 0 = A - \frac{B}{b^2}$$

$$\Rightarrow \quad A = \frac{B}{b^2} \quad \& \quad B = \frac{pa^2b^2}{b^2 - a^2}, \quad A = \frac{pa^2}{b^2 - a^2}$$

$$\sigma_r = \frac{pa^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) = \frac{-p}{1 - \frac{b^2}{a^2}}\left(1 - \frac{b^2}{r^2}\right)$$

$$\sigma_\theta = \frac{-p}{1 - \frac{b^2}{a^2}}\left(1 + \frac{b^2}{r^2}\right)$$

$$\varepsilon_\theta = \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right) \quad \because \sigma_z = 0$$

$$\Rightarrow \quad \varepsilon_\theta = \frac{u}{r} = -\frac{p}{E\left(1-\frac{b^2}{a^2}\right)}\left[1+\frac{b^2}{r^2} - \nu\left(1-\frac{b^2}{r^2}\right)\right]$$

$$\Rightarrow \quad \frac{u(a)}{a} = \frac{p}{E}\left[\frac{b^2+a^2}{b^2-a^2} + \nu\right]$$

(b) Balance of forces gives a radial pressure $p/2$ on solid cylinder & p on annulus.

Solid cylinder $\quad \sigma_r = \sigma_\theta = -\frac{p}{2}$

$$\varepsilon_\theta(a) = \frac{u_c(a)}{a} = -\frac{p}{2E}(1-\nu)$$

Interface compatibility $\Rightarrow$

$$u(a) = u_c(a) + \delta$$

$$\Rightarrow \quad p\left[\frac{b^2+a^2}{b^2-a^2} + \nu\right] = -\frac{p}{2}(1-\nu) + \frac{\delta E}{a}$$

For $b/a = 2$ & $\nu = \frac{1}{3}$ $\quad p = \frac{3}{7}\frac{\delta E}{a}$

(c) Yield initiates at inner radius a of annulus. The active Tresca surface is

$$\sigma_\theta - \sigma_r = -\frac{2p\, b^2/r^2}{1 - b^2/a^2} = Y$$

at $r = a$ $\quad p = \frac{3Y}{8} = \frac{3\delta E}{7a}$

$$\Rightarrow \frac{\delta}{a} = \frac{7}{8}\frac{Y}{E}$$

(d) This is an overestimate of the allowable since it neglects the out-of-plane shear stresses due to the difference in thickness between the annulus & solid cylinder.

3.

(a) $$\phi = Br^n\theta$$

$$\nabla^2\phi = \left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)$$

$$= B\left[n(n-1) + n\right] r^{n-2}\theta$$

$$= B n^2 r^{n-2}\theta$$

$$\nabla^4\phi = B\left[(n-2)(n-3) + (n-2)\right] n^2 r^{n-4}\theta = 0$$

$$\Rightarrow B n^2 (n-2)^2 r^{n-4}\theta = 0$$

$$n = (0, 2).$$

(b) $$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi_2}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi_2}{\partial\theta^2} = 2B_2\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi_2}{\partial r^2} = 2B_2\theta$$

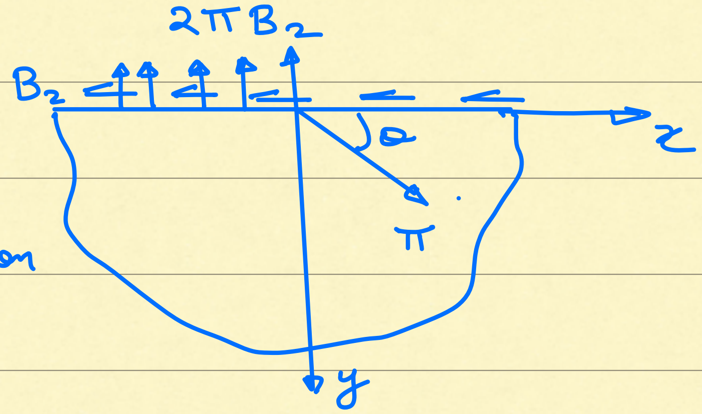
$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi_2}{\partial\theta}\right) = -B_2$$

$$\Rightarrow \sigma_{\theta\theta}(r,0) = 0, \quad \sigma_{r\theta}(r,0) = -B_2$$

$$\sigma_{\theta\theta}(r,\pi) = 2\pi B_2 \quad \& \quad \sigma_{r\theta}(r,\pi) = -B_2$$

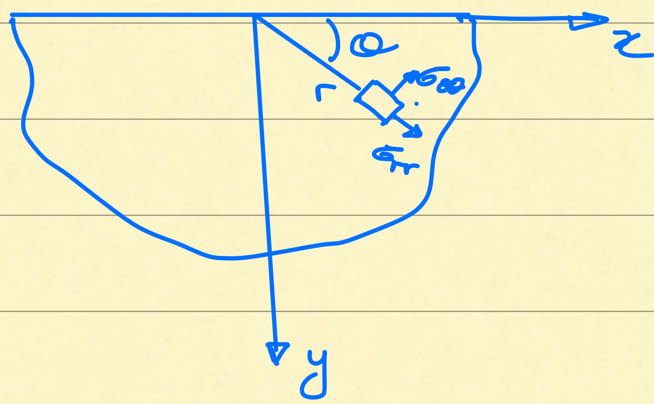
$\Rightarrow$

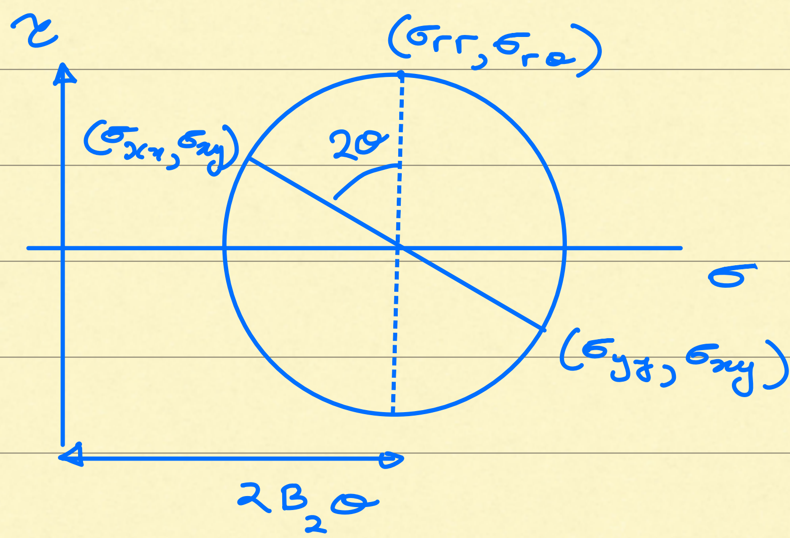
ie surface $y=0$ loaded with uniform shear traction $-B_2$ & normal traction $2\pi B_2$ on $x<0$.



(c)

Converting to Cartesian co-ordinates





$$\sigma_{xx} = B_2(2\theta - \sin 2\theta)$$

$$\sigma_{yy} = B_2(2\theta + \sin 2\theta)$$

$$\sigma_{xy} = -B_2 \cos 2\theta$$

Recalling $\sin 2\theta = 2\sin\theta\cos\theta$

ie $$\sigma_{xx} = B_2\left[2\tan^{-1}\left(\frac{y}{x}\right) - \frac{2xy}{x^2+y^2}\right]$$

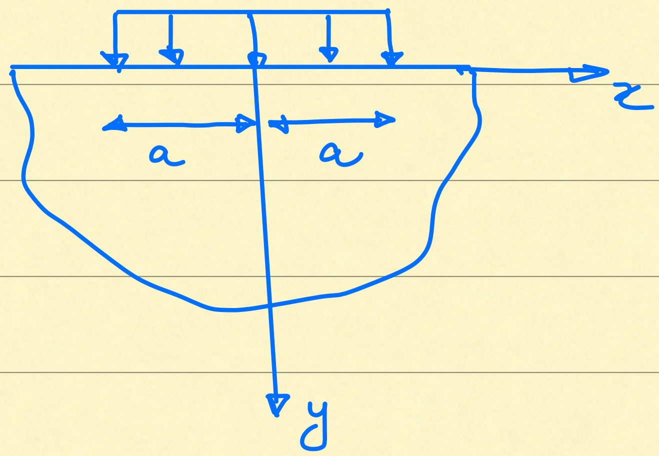
$$\sigma_{yy} = B_2\left[2\tan^{-1}\left(\frac{y}{x}\right) + \frac{2xy}{x^2+y^2}\right]$$

& $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$\Rightarrow \sigma_{xy} = -B_2\left[\frac{x^2-y^2}{x^2+y^2}\right]$$

(d)

The problem under consideration now is



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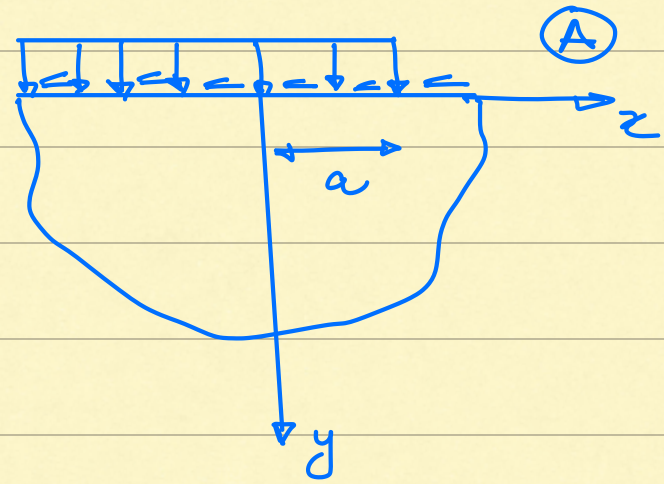

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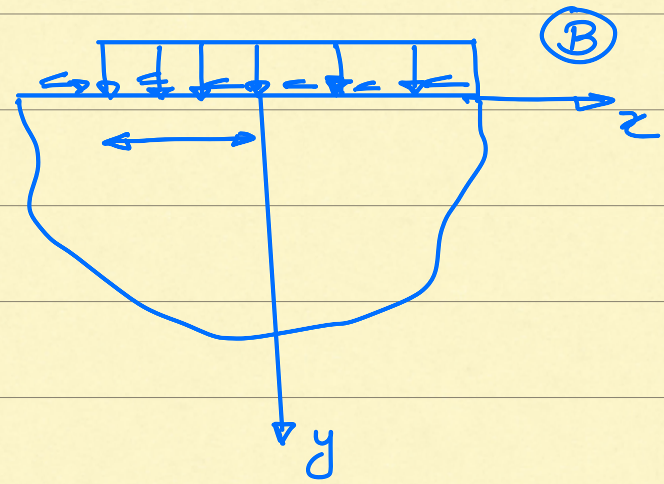


Shear stress in A

$$\sigma_{xy}^{A} = -B_2\left[\frac{(x-a)^2-y^2}{(x-a)^2+y^2}\right]$$

Shear stress in B

$$\sigma_{xy}^{B} = -B_2 \left[\frac{(x+a)^2 - y^2}{(x+a)^2 + y^2} \right]$$

$$\Rightarrow \sigma_{xy} = \sigma_{xy}^{A} - \sigma_{xy}^{B}$$

$$= -B_2 \left[\frac{(x-a)^2 - y^2}{(x-a)^2 + y^2} - \frac{(x+a)^2 - y^2}{(x+a)^2 + y^2} \right]$$

$$\sigma_{yy}(x,0) = 2\pi B_2 \quad \text{on} \quad -a \leq x \leq a$$

$$= 0 \qquad |x| > a$$

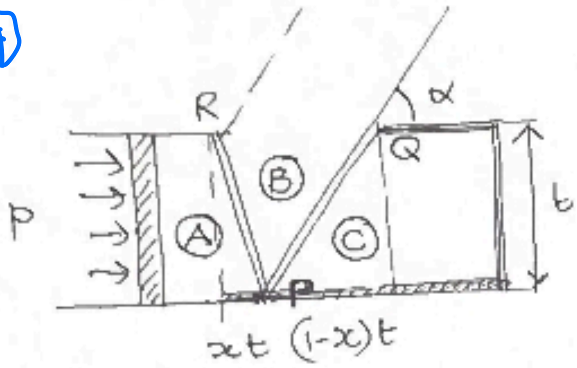
$$\Rightarrow 2\pi B_2 = -p \qquad \therefore B_2 = -\frac{p}{2\pi}$$

$$\sigma_{xy}(x,y) = \frac{p}{2\pi} \left[\frac{(x-a)^2 - y^2}{(x-a)^2 + y^2} - \frac{(x+a)^2 - y^2}{(x+a)^2 + y^2} \right]$$

$$\& \; \sigma_{xy}(x,a) = \frac{p}{2\pi}\left[\frac{(x-a)^2-a^2}{(x-a)^2+a^2} - \frac{(x+a)^2-a^2}{(x+a)^2+a^2}\right]$$

$$= -\frac{p}{2\pi} \; \frac{4a^3x}{(x^2+2a^2)^2-4a^2x}$$

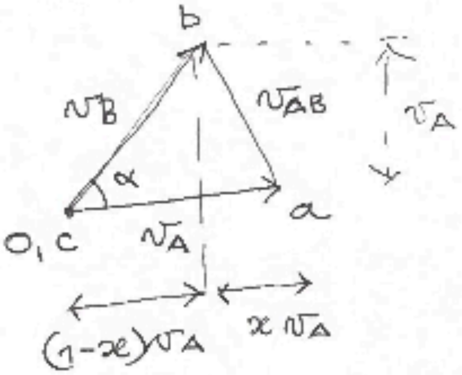
4



$$\overline{PQ} = t\sqrt{(1-x)^2+1} = t\sqrt{x^2-2x+2}$$

$$\overline{PR} = t\sqrt{1+x^2}$$

HODOGRAPH



$$v_B = v_A\sqrt{x^2-2x+2}$$

$$v_{AB} = v_A\sqrt{1+x^2}$$

$$W_{ext} = p \cdot t \cdot v_A$$

$$W_{int} = k\left[v_B \cdot \overline{PQ} + v_{AB} \cdot \overline{PR} + f\,x\,t\right] =$$

$$= k\,t\,v_A\left[x^2-2x+2+1+x^2+f\,x\right] =$$

$$= k\,t\,v_A\left[2x^2-(2-f)x+3\right]$$

$$W_{ext} = W_{int}$$

$$p\cancel{t} \cdot \cancel{v_A} = k\cancel{t}\,\cancel{v_A}\left[2x^2-(2-f)x+3\right]$$

$$p = k\left[2x^2-(2-f)x+3\right]$$

$$\frac{\partial p}{\partial t} = 0 \qquad k\left[4x-(2-f)\right] = 0 \Rightarrow x = \frac{2-f}{4}$$

(i) if $f = 0$

$x = \frac{1}{2}$ $\quad (1-x) = \frac{1}{2}$ $\quad \tan\alpha = \frac{2\cancel{t}}{\cancel{t}} = 2$

hence $\alpha = \tan^{-1}(2)$

(ii) if $f = 0.2$ $\quad x = \frac{1.8}{4} = 0.45$

(iii) $F = pt = kt\left[\cancel{2}^{1} \cdot \frac{(2-f)^2}{\cancel{16}\,8} - \frac{(2-f)^2}{4} + 3\right] =$

$F = kt\left[3 - \frac{(2-f)^2}{8}\right] = 2.68\ kt$