3A6 Final Exam - Crib

Q1. Flow through a channel

(a) The boundary layer grows as $x^{0.5}$. The behaviour of the boundary layer thickness is proportional to the diffusivities.

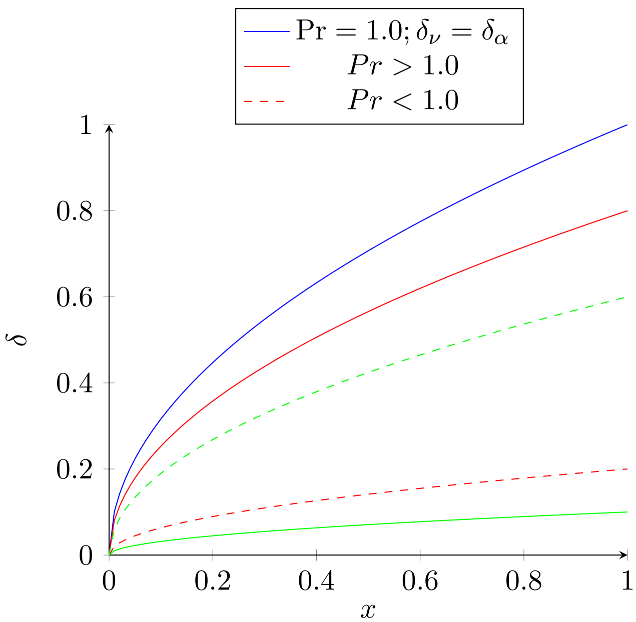


Figure 1: Schematic of evolution of δ_ν (solid) and δ_α (dashed), for values of Pr as shown in the legend. Not to scale.

(b)

(i)

$$\rho c_p \mathbf{u} \cdot \nabla T = \lambda \nabla^2 T$$
$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

We have $v = 0$ and $\dfrac{\partial^2 T}{\partial x^2} \approx \mathcal{O}(\Delta T L^2)$, while $\dfrac{\partial^2 T}{\partial x^2} \approx \mathcal{O}(\dfrac{\Delta T}{H^2})$. Therefore, for $H << L$ we can neglect $\dfrac{\partial^2 T}{\partial x^2}$. The final equation is then:

$$U_b\frac{\partial T}{\partial x} = \alpha\left(\frac{\partial^2 T}{\partial y^2}\right)$$

which can be nondimensionalized by $(T_\infty - T_0)$ to yield:

$$U_b\frac{\partial \theta}{\partial x} = \alpha\left(\frac{\partial^2 \theta}{\partial y^2}\right)$$

(ii) Here the only request was to show by substitution, but some have also directly solved the equation, so either are fine. By substitution, we have:

$$
\begin{aligned}
\theta &= \exp\left(- \frac{\pi^2}{4\ \mathrm{Pe}} \frac{x}{H} \right) \cos\left(\frac{\pi}{2} \frac{y}{H} \right) \\
\frac{\partial \theta}{\partial x} &= -\frac{\pi^2}{4\ \mathrm{Pe} H} \exp\left(- \frac{\pi^2}{4\ \mathrm{Pe}} \frac{x}{H} \right) \cos\left(\frac{\pi}{2} \frac{y}{H} \right) \\
\frac{\partial \theta}{\partial y} &= \frac{\pi}{2\ H} \exp\left(- \frac{\pi^2}{4\ \mathrm{Pe}} \frac{x}{H} \right) \sin\left(\frac{\pi}{2} \frac{y}{H} \right) \\
\frac{\partial^2 \theta}{\partial y^2} &= -\frac{\pi^2}{4\ H^2} \exp\left(- \frac{\pi^2}{4\ \mathrm{Pe}} \frac{x}{H} \right) \cos\left(\frac{\pi}{2} \frac{y}{H} \right)
\end{aligned}
$$

Substituting in the governing equation, we have:

$$
U_b \left[-\frac{\pi^2}{4\ \mathrm{Pe} H} \exp\left(- \frac{\pi^2}{4\ \mathrm{Pe}} \frac{x}{H} \right) \cos\left(\frac{\pi}{2} \frac{y}{H} \right) \right] = \alpha \left[-\frac{\pi^2}{4\ H^2} \exp\left(- \frac{\pi^2}{4\ \mathrm{Pe}} \frac{x}{H} \right) \cos\left(\frac{\pi}{2} \frac{y}{H} \right) \right]
$$

$$
\frac{U_b}{\mathrm{Pe}} = \frac{\alpha}{H}
$$

$$
\boxed{\mathrm{Pe} = \frac{U_b H}{\alpha}}
$$

This is a Peclet number, representing the convection of energy relative to its molecular diffusion.

(iii) The mean non-dimensional temperature is given as:

$$
\begin{aligned}
\theta_b &= \frac{1}{2\ H} \int_{-H}^{+H} c\ dy \\
&= \left[\frac{1}{2\ H} \exp\left(- \frac{\pi^2}{4\ \mathrm{Pe}} \frac{x}{H} \right) \frac{2}{\pi\ H} \sin\left(\frac{\pi}{2} \frac{y}{H} \right) \right]_{-H}^{H}
\end{aligned}
$$

$$
\boxed{\theta_b = \frac{2}{\pi} \exp\left(- \frac{\pi^2}{4\ \mathrm{Pe}} \frac{x}{H} \right)}
$$

(iv) The heat transfer rate per unit length is calculated from :

$$
\begin{aligned}
q = -\lambda \frac{\partial T}{\partial x}\bigg|_{\pm H} &= \lambda (T_\infty - T_0) \frac{\partial \theta}{\partial y}\bigg|_{\pm H} \\
&= \lambda (T_\infty - T_0) \frac{\pi}{2\ H} \exp\left(- \frac{\pi^2}{4\ \mathrm{Pe}} \frac{x}{H} \right) \left[\sin\left(\frac{\pi}{2} \frac{y}{H} \right) \right]_{\pm H} \\
q = \lambda (T_\infty - T_0) \frac{\pi}{2\ H} \exp\left(- \frac{\pi^2}{4\ \mathrm{Pe}} \frac{x}{H} \right)
\end{aligned}
$$

The Nusselt number can therefore be calculated as:

$$Nu = \frac{qH}{\lambda(T_b - T_0)} = H\lambda(T_\infty - T_0)\frac{\pi}{2\ H}\exp\left(-\frac{\pi^2}{4\ \mathrm{Pe}}\frac{x}{H}\right)\frac{1}{(T_\infty - T_0)\frac{2}{\pi}\exp\left(-\frac{\pi^2}{4\ \mathrm{Pe}}\frac{x}{H}\right)}$$

$$\boxed{Nu = \frac{\pi^2}{4}}$$

Q2. Mass diffusion

(a) In this case we have unsteady diffusion into a container with uniform composition. The mass balance for species A in the liquid is:

$$\begin{aligned}
\frac{d}{dt}(c_l V) = j_A S - w_A V = h_g S(c_{g,\infty} - c_{g,0}) - k_R c_l = \\
\frac{dc_l}{dt} = \frac{h_g S}{V}(c_{g,\infty} - c_{g,0}) - k_R c_l \\
\frac{dc_l}{dt} = \frac{h_g S}{V}(c_{g,\infty} - K c_l) - k_R c_l \\
\frac{dc_l}{dt} + (k_R + \frac{h_g S K}{V}) c_l = \frac{h_g S}{V} c_{g,\infty}
\end{aligned}$$

which can be solved as:

$$\boxed{c_l = \frac{h_g S}{h_g S + k_R V} c_{g,\infty}(1 - \exp[-(\tfrac{h_g S}{V} + k_R)t)])}$$

(b)
(i) For a column of liquid, we have a balance of the change in the flux and the reaction rate per unit volume:

$$\frac{\partial c_l}{\partial t} = \frac{j_A}{dx} - w_A = D\frac{\partial^2 c_l}{\partial x^2} - k_R c_l$$

(ii) Under steady state we have:

$$\begin{aligned}
D\frac{\partial^2 c_l}{\partial x^2} - k_R c_l = 0 \\
c_l = A e^{\sqrt{k_R/D}\, x} + B e^{-\sqrt{k_R/D}\, x}
\end{aligned}$$

The concentration at $x \to \infty$ is zero, so that $A = 0$, and $c_l = B e^{-\sqrt{k_R/D}\, x}$. This also implies that $B = c_{l,0}$. At the gas-liquid boundary, we have a convective flux $j_A = h(c_{g,\infty} - c_{g,0}) = h(c_{g,\infty} - K c_{l,0})$, so that:

$$\begin{aligned}
j_A = h(c_{g,\infty} - K c_{l,0}) = -D\frac{\partial c_l}{\partial x}\bigg|_{x=0} \\
h(c_{g,\infty} - K c_{l,0}) = c_{l,0} D\sqrt{\frac{k_R}{D}} \\
c_{l,0} = \frac{h c_{g,\infty}}{\sqrt{k_R D} + hK} \\
c_{l,0} = \frac{c_{g,\infty}}{\frac{\sqrt{k_R D}}{h} + K}
\end{aligned}$$

Note that for very slow reaction relatively to convection, the concentration in the liquid at the interface equals the equilibrium concentration corresponding to $c_{g,\infty}$. The final result for the concentration of A in the liquid is therefore:

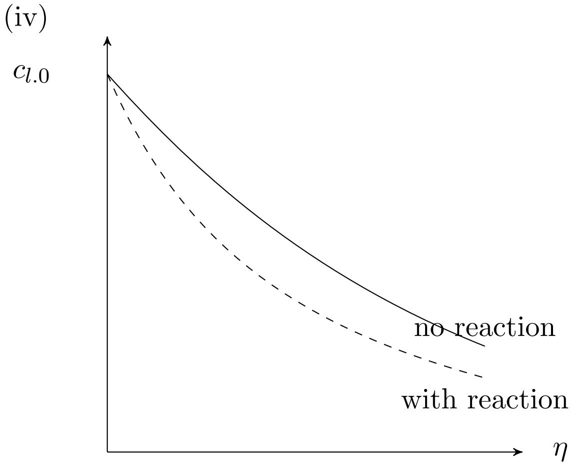
$$\boxed{c_l = \frac{c_{g,\infty}}{\dfrac{\sqrt{k_R D}}{h} + K} \exp(-\sqrt{k_R/D}\; x)}$$

(iii) By substitution, we have, using $\eta = x/\sqrt{4Dt}$:

$$\begin{aligned}
\frac{\partial c_l}{\partial t} &= c_{l,0} \exp(-\eta^2)(-2\eta)\frac{\partial \eta}{\partial t} \\
\frac{\partial c_l}{\partial x} &= c_{l,0} \exp(-\eta^2)(-2\eta)\frac{\partial \eta}{\partial x} \\
\frac{\partial^2 c_l}{\partial x^2} &= c_{l,0} \exp(-\eta^2)(-2\eta)^2 \left(\frac{\partial \eta}{\partial x}\right)^2 + c_{l,0} \exp(-\eta^2)(-2\eta)\frac{\partial^2 \eta}{\partial x^2} \\
\frac{\partial \eta}{\partial t} &= -\frac{1}{2}\frac{x t^{-3/2}}{\sqrt{4Dt}} = -\frac{\eta}{2t} \\
\frac{\partial c_l}{\partial x} &= \frac{\eta}{x} \\
\frac{\partial^2 \eta}{\partial x^2} &= 0 \\
D\frac{\partial^2 c_l}{\partial x^2} - \frac{\partial c_l}{\partial x} &= c_{l.0} \exp(-\eta^2)\left[D(-2\eta)^2\left(\frac{\eta}{x}\right)^2 - (-2\eta)(-\frac{\eta}{2t})\right] = \\
&= c_{l.0} \exp(-\eta^2)\frac{1}{t}\left[\frac{4Dt}{x^2}\eta^2\eta^2 - \eta^2\right] = 0
\end{aligned}$$

Q.E.D.

(iv)



Q3. Fin

(a) Steady state energy balance along an element dx, with conduction and convection along perimeter P:

$$-\lambda S\frac{dT}{dx} + S\left(\lambda\frac{dT}{dx} + \lambda\frac{d^2T}{dx^2}\right) + hP(T_\infty - T) = 0$$
$$\frac{d^2T}{dx^2} - \frac{hP}{\lambda S}(T - T_\infty) = 0$$

The general solution is therefore:

$$\theta = Ae^{mx} + Be^{-mx}$$

where $m = \sqrt{\dfrac{hP}{\lambda S}}$.

For the hot side, $x < 0$:

$$T_0 - T_h = A + B$$
$$0 = Ae^{-mL_h} + Be^{mL_h}$$
$$B = -Ae^{-2mL_h}$$
$$T_0 - T_h = A - Ae^{-2mL_h} = A(1 - e^{-2mL_h}) \to A = \frac{T_0 - T_h}{1 - e^{-2mL_h}}$$
$$T - T_h = (T_0 - T_h)\frac{e^{mx} - e^{-2mL_h}e^{-mx}}{1 - e^{-2mL_h}} = (T_0 - T_h)\frac{\sinh[m(x + L_h)]}{\sinh[mL_h]}$$

And for the cold side, $x > 0$::

$$T_0 - T_c = C + D$$
$$0 = Ce^{mL_c} + De^{-mL_c} \to D = -Ce^{2mL_c}$$
$$T_0 - T_c = C(1 - e^{2mL_c}) \to C = \frac{T_0 - T_c}{1 - e^{2mL_c}}$$
$$T - T_c = (T_0 - T_c)\frac{e^{mx} - e^{2mL_c}e^{-mx}}{1 - e^{2mL_c}} = -(T_0 - T_h)\frac{\sinh[m(x - L_c)]}{\sinh[mL_c]}$$

(b) For continuity of heat transfer across $x = 0$:

$$-\lambda\frac{dT}{dx}\bigg|_{x=0^-} = -\lambda\frac{dT}{dx}\bigg|_{x=0^+} (T_0 - T_h)m\frac{1 + e^{-2mL_h}}{1 - e^{2mL_h}} = (T_0 - T_c)m\frac{1 + e^{2mL_c}}{1 - e^{2mL_c}}$$
$$(T_0 - T_h)\frac{1}{\tanh mL_h} = -(T_0 - T_c)\frac{1}{\tanh(mL_c)}$$
$$T_0 = \frac{\dfrac{T_h}{\tanh(mL_h)} + \dfrac{T_c}{\tanh(mL_c)}}{\dfrac{1}{\tanh(mL_c)} + \dfrac{1}{\tanh(mL_h)}}$$

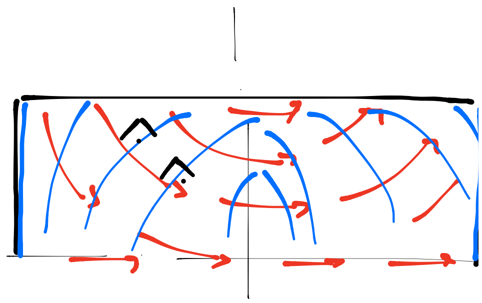


Figure 2: Sketch symmetry top half for $L_h = L_c$ isotherms (blue) and isoflux (red) lines

(c) A suitable Biot number in this case characterizes the cross-fin heat transfer along a dimension $R \approx P/S$ characteristic of a dimension along its cross section (not the heat transfer along the fin, otherwise it would not work as a fin!):

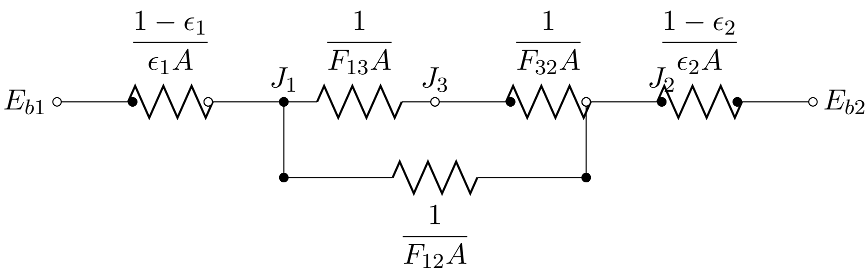
$$Bi \approx \frac{h}{\lambda/R} \approx \frac{hS}{\lambda P}$$

so that $Bi << 1$ (a thin rod) means a uniform temperature across the cross section. Appropriate answers would be those giving a similar order of magnitude. Note that $\frac{hP}{\lambda S}L^2$ does not give the right order of magnitude for this purpose, as it is of course the leading order term for convection along (not across) the fin.

(d) For larger Biot numbers, conduction across the rod becomes of the same order as along the rod (and convection), so that the temperature is no longer uniform. The boundary conditions are still given, so the job here is to sketch out lines that are isothermal and the perpendicular lines for isofluxes, while watching out for the symmetry lines along the rod. In addition, the heat flux at the isothermal ends is zero, because $q = h(T_h - T_h) = h(T_c - T_c) = 0$.

Q4. Radiation in a channel

(a) The radiation circuit is shown below:



The form factors $F_{11} = F_{22} = F_{33} = 0$. The following relations are valid:

$$F_{12} + F13 = 1$$
$$F_{21} + F23 = 1$$
$$F_{31} + F32 = 1$$

From symmetry, $F_{ij} = F_{ji}$, and therefore $F_{12} = F_{13} = F_{23} = 0.5$.

We can solve for the circuit, to get an equivalent resistance. The net heat transfer is therefore:

$$Q_1 = \frac{(E_{b1} - E_{b2})A}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{4}{3} + \frac{1-\epsilon_2}{\epsilon_2}} = (5.67 \times 10^{-8}\ \mathrm{W/m^2K^{-4}}) \frac{(1000^4 - 7000^4)\ \mathrm{K^4}(1\ \mathrm{m^2/m})}{\frac{1-0.33}{0.33} + \frac{4}{3} + \frac{1-0.5}{0.5}}$$

$$Q_1 = 9943\ \mathrm{W/m}$$

(b) Since there is no net radiative heat transfer to surface 3, $E_{b3} = J_3$. We solve the circuit for J_3. The resistance through the 1-3 path is twice that of the 1-2 path, so that the heat transfer through the former is 1/3 of the total: $Q_{13} = Q_1/3$. We can therefore write:

$$E_{b3} = E_{b1} = Q_1 \frac{1-\epsilon_1}{\epsilon_1 A} - \frac{1}{2} Q_1 \frac{2}{A}$$

which can be solved for T_3 using $E_{b3} = \sigma T_3^4$:

$$E_{b3} = (5.67 \times 10^{-8}\ \mathrm{W/m^2K^{-4}})(1000\ \mathrm{K^4}) - (9943 \mathrm{W/m^2})(2 + \frac{2}{3}) = 30185\ \mathrm{W/m^2}$$

from which

$$T_3 = 864\ \mathrm{K}$$

Since there is no heat transfer from 3, the emissivity of the surface has no effect on the overall result.

(c) Assume that there is also convection in the channel. For each surface, the rate of heat transfer is changed by a rate of convection. For surface 1, we have:

$$Q_1 = \frac{(E_{b1} - E_{b2})A}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{4}{3} + \frac{1-\epsilon_2}{\epsilon_2}} + h(T_1 - T_b) = 1.5Q_{1,0}$$

This produces a non-linear equation for T_1, which can be solved iteratively:

$$f(T) = \frac{(5.67 \times 10^{-8}\ \mathrm{W/m^2K^{-4}})T_1^4 - 13614\ \mathrm{W/m^2}}{4.33} + 10\ \mathrm{W/m^2}(T_1 - T_b) - 1.5(9943)\ \mathrm{W/m^2} = 0$$

T	f(T)
1000	2237
900	-3266
950	-692
970	+434
960	-135
965	97

so the solution is somewhere around 960 and 965 K. The corresponding heat transfer rate is obtained as:

$$Q_2 = \frac{E_{b1} - E_{b2}}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{4}{3} + \frac{1-\epsilon_2}{\epsilon_2}} - h(T_2 - T_b) = 1361\ \mathrm{W/m}$$