Module 4C6 solutions 2017

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Q1. (a) (i) Hammer:

Advantages: Simple, reasonably cheap, requires no reference fixture, easily moved around to measure different points, adds no mass or damping to the structure once it has bounced off, bandwidth can be tailored (within limits) by choice of tip and mass.

Disadvantages: Hard to get large amplitudes, so may not work very well in a noisy environment, impacts can damage a fragile structure, difficult to get to very high frequencies because it is hard to get very short impacts (especially on light structures). Signal-to-noise ratio can be poor, because amplifier gain has to be set to catch the high peak force without clipping.

Shaker:

Advantages: allows a wide range of different input signals (sine, noise, chirp etc), relatively easy to achieve high force levels, easier to get good signal-to-noise level.

Disadvantages: needs a fixed reference to "push against", needs an attachment to the structure and hence adds mass and damping, easy to make mistakes in set-up (e.g. off-axis loading), difficult to move to a different test position.

(ii) Accelerometer:

Advantages: relatively small, relatively cheap (very cheap for MEMS types), easy to set up, doesn't need a fixed reference.

Disadvantages: needs to be attached to the structure so it adds mass and perhaps damping (especially via the cable), trade-off between bandwidth and signal level because you need a small device to reach high frequencies.

Laser vibrometer:

Advantages: no contact so no added mass or damping, can be used for inaccessible measurements (e.g. pinnacles of a building, through a window to see inside a furnace).

Disadvantages: very expensive, fiddly and temperamental to set up, may need to modify the structure to obtain sufficiently good reflection, needs a fixed reference (e.g. firm ground to stand a tripod).

(iii) Hammer/laser: good for lightweight structures such as a violin body. No added mass or damping apart from a small piece of reflective tape. Probably want to choose a very small impulse hammer for this application.

Hammer/accelerometer: good for field testing because both are easily portable and don't need a fixed reference or complicated set-up. Example: quality control of car bodies coming off a production line.

(b) (i) The first and second modes have similar peak amplitudes, but the frequency response shows no antiresonances so the signs must alternate. We see two right-facing circles and one left-facing circle, so the small right-facing circle is mode 1, the left-facing circle is mode 2, and the larger right-facing circle is mode 3.

(ii) Measure frequencies from Fig. 1, and find half-power bandwidths by counting the points round a semi-circle for each mode, knowing that the spacing is 0.5 Hz. Estimate peak amplitudes from circle diameter, then calculate the mode-shape product by using the fact that the amplitude at a single isolated peak is approximately

$$\frac{i\omega a_n}{2i\omega\omega_n\zeta_n} = \frac{a_n Q_n}{\omega_n}$$

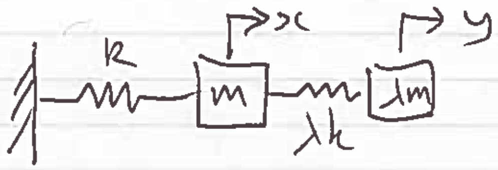
using a single term of the formula from section 7 of the Data Sheet with $\omega = \omega_n$. A typical set of results is:

Mode	f_n (Hz)	δf (Hz)	Q_n	a_n
1	135	2.5	54	0.94
2	170	2.5	68	-0.94
3	280	3	93	2.17

The actual values used in the simulation program were

Mode	f_n (Hz)	Q_n	a_n
1	135.3	50	1
2	170.7	70	-0.9
3	280.5	100	2

2 (a)



Kinetic energy $T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \lambda m \dot{y}^2$

Potential energy $V = \frac{1}{2} k x^2 + \frac{1}{2} \lambda k (x-y)^2$

So $M = m \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$, $K = k \begin{bmatrix} 1+\lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix}$

Natural frequencies satisfy $\begin{vmatrix} 1+\lambda-\Omega^2 & -\lambda \\ -\lambda & \lambda-\lambda\Omega^2 \end{vmatrix} = 0$

where $\Omega^2 = \omega^2 \frac{m}{k}$

$\therefore (1-\Omega^2)(1+\lambda-\Omega^2) = \lambda$

$\therefore \Omega^4 - (2+\lambda)\Omega^2 + 1 = 0$

$\therefore \Omega^2 = \frac{1}{2}\left[2+\lambda \pm \sqrt{(2+\lambda)^2 - 4}\right]$

$= 1 + \frac{\lambda}{2} \pm \sqrt{\lambda + \lambda^2/4}$

$\simeq 1 \pm \sqrt{\lambda}$ for $\lambda << 1$

Mode shapes: $k(-\lambda x + \lambda y) = \omega^2 m \lambda y$

$\therefore \quad y - x = \Omega^2 y \rightarrow x = (1-\Omega^2) y \simeq \mp\sqrt{\lambda}\, y$

(b) Replace λk by $\lambda k(1+i\eta)$. The Rayleigh quotient gives an approximation to the complex value of ω^2 provided a reasonable approximation to the mode shape is used, and if $\eta << 1$ we can use the undamped modes from (a) for this.

So $\bar{\omega}^2 \simeq \dfrac{k\left[x^2 + \lambda(1+i\eta)(x-y)^2\right]}{m(x^2+\lambda y^2)}$

complex values

$\therefore \bar{\Omega}^2 \simeq \dfrac{x^2 + \lambda(1+i\eta)(x-y)^2}{x^2+\lambda y^2}$

So if $x = \mp\sqrt{\lambda}\, y$, $\bar{\Omega}^2 \simeq \frac{1}{2}\left[1 + (1+i\eta)(1\pm\sqrt{\lambda})^2\right]$

2 cont

$$\text{so} \quad \bar{\Omega}^2 \simeq \tfrac{1}{2}\left[1 + (1 \pm i\eta)(1 \pm 2\sqrt{\lambda})\right]$$

$$\simeq \tfrac{1}{2}\left(1 + 1 + i\eta \pm 2\sqrt{\lambda}\right) = 1 \pm \sqrt{\lambda} + \frac{i\eta}{2}$$

assuming λ and η both small.

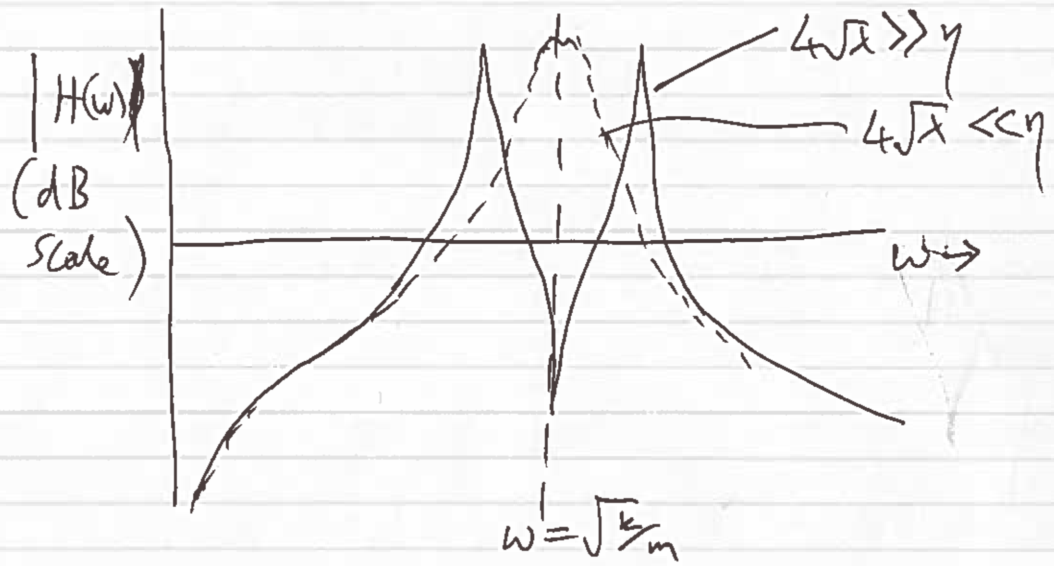
$$\text{Now} \quad Q_n \simeq \frac{\mathrm{Re}(\bar{\omega}_n^2)}{\mathrm{Im}(\bar{\omega}_n^2)} = \frac{\mathrm{Re}(\bar{\Omega}_n^2)}{\mathrm{Im}(\bar{\Omega}_n^2)} \quad \text{(datasheet}$$

$$\therefore \quad Q_n \simeq \frac{2(1 \pm \sqrt{\lambda})}{\eta} \simeq \frac{2}{\eta} \quad \text{for both modes.}$$

(c) Modal overlap factor $c \simeq \dfrac{\tfrac{1}{2}\text{-power bandwidth}}{\text{spacing}}$

$$\simeq \frac{\omega_n/Q_n}{\omega_2 - \omega_1} \simeq \frac{1/Q_n}{2\sqrt{\lambda}} = \frac{\eta}{4\sqrt{\lambda}}$$

So if $\begin{cases} 4\sqrt{\lambda} \gg \eta & \text{peaks are well separated} \\ 4\sqrt{\lambda} < \eta & \text{peaks overlap and merge} \end{cases}$

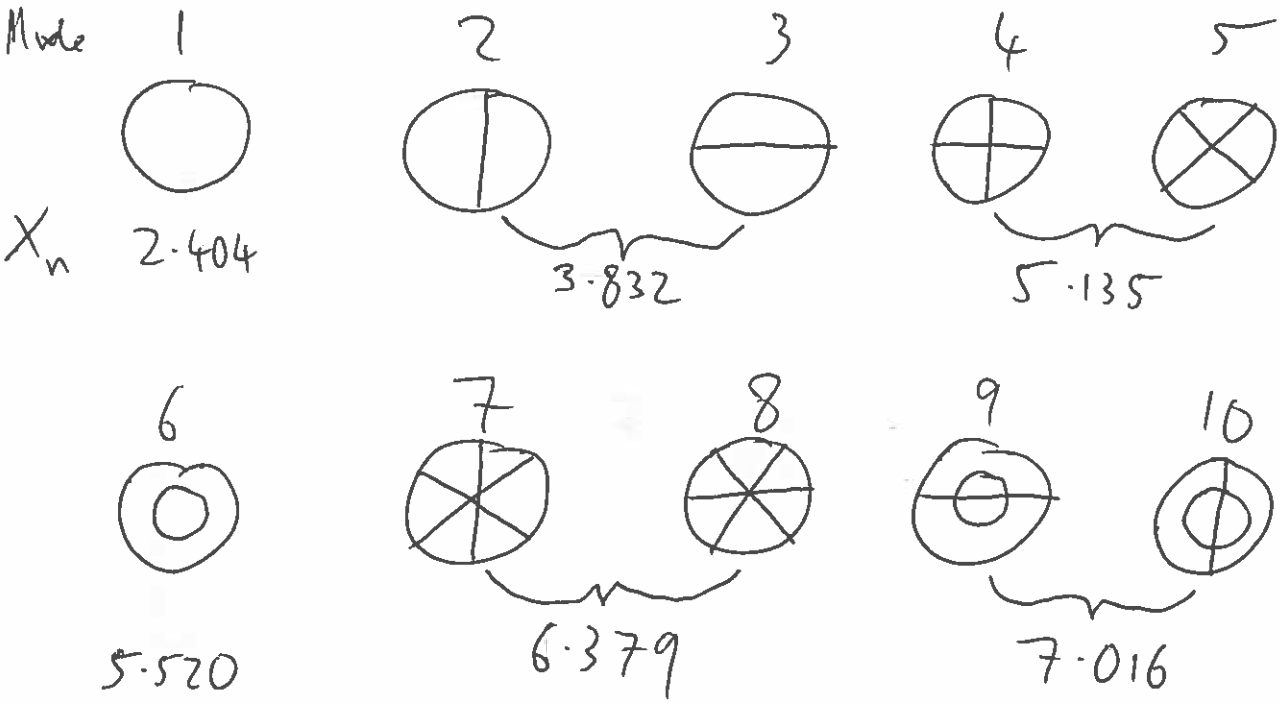


(d) The smaller the value of λ the bigger the motion of λm compared to m. In practice there will be a maximum amplitude before it hits end stops or some other nonlinearity happens. So want the smallest λ you can get away with given this limitation.

Q3 (a) From Data Sheet, frequency ω_n is given by

$$\omega_n = \frac{X_n}{a}\sqrt{\frac{T}{m}}$$

where X_n is the dimensionless number tabulated in the Data Sheet. Noting that all modes except the axisymmetric ones associated with Bessel function J_0 appear in pairs, the first ten modes are as follows, with their values of X_n.

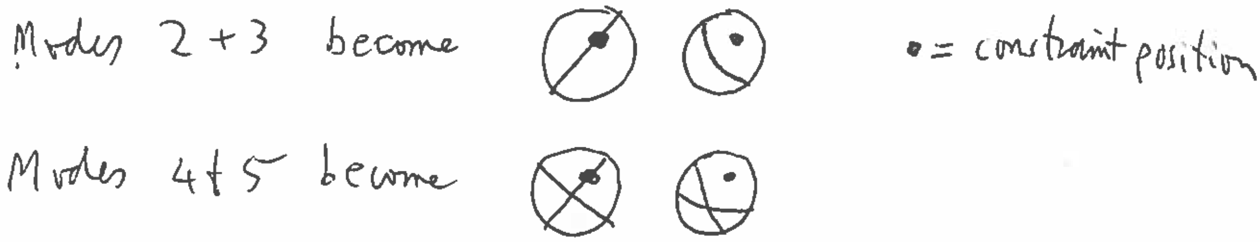


(b) For any given mode, if ω_n is to be increased by a factor 2 purely by changing a, then a must be halved. Alternatively, if a is kept fixed then $\sqrt{T/m}$ must be doubled. That would mean increasing tension by a factor 4, decreasing m by a factor 4, or some combination of the two. Any of these options is likely to be very problematic in practice. Modern drums use polymer membranes, and there is only a limited range of tension within which the behaviour is satisfactory. There will be a limit beyond which the membrane fails, but if the tension is too low the drummer will find it very "soggy" and unsatisfactory. Drums usually operate not very far below the failure stress. (Similar considerations apply to stretched string used for musical instruments.) Similarly, to decrease the mass per unit area by factor 4 with a given polymer would mean reducing the thickness by the same factor, increasing the stress for a given tension and risking failure.

(c) Applying a single point constraint will produce a new set of frequencies that interlace the old ones, moving them in an upward direction. Since many of the original frequencies appeared in pairs, for any single constraint there must still be at least one frequency matching each of those pairs.

(i) For a central constraint, only the axisymmetric modes are changed. All the others have a nodal point at the centre and are unaffected. So only natural frequencies 1 and 6 change. They both move up, but cannot go higher than the frequencies of modes 2 and 7 respectively. Sketches of the expected mode shapes are as follows:

(ii) For an off-centre constraint position, not matching the radius of any of the relevant nodal circles, all modes are affected. From each pair, the one that stays fixed will have a radial node line through the constrained point, while the other of the pair will shift to a higher frequency, but never higher than the next original natural frequency. Example sketches:



(d) If a mass is used in place of a constraint, the arguments about which modes are an are not changed remain valid. However, now the frequencies all move downwards: the problem can be seen as point coupling of the original membrane to a free mass, which had a "natural frequency" of zero.

Q4 (a) The transient decay rate (log dec) of individual resonant modes can be measured provided the decay is slow enough that a large number of cycles can be detected. That makes the method good for materials with low damping, and increasingly accurate as damping gets lower. It would not be accurate for materials with higher damping, giving insufficient cycles before decay below the noise floor of the measurement. The geometry of the test specimen needs to be designed to have well-separated modes: low modal overlap is needed so that decay rates of different modes can be separated, by some kind of band-pass filtering approach. Care is always needed when measuring low damping, because extra damping is likely to be contributed by whatever is used to support the test specimen, any attached sensors, and losses by viscosity or sound radiation into the surrounding air.

Frequency domain methods based on transfer functions, e.g. by circle fitting, are the second possibility. This method is difficult for very low damping because you need to collect data for a long time to get enough points around the modal circle to obtain a good fit. The same is true of a half-power bandwidth measurement: it is easier to get an accurate bandwidth of a moderately wide peak. But if damping is too high then a clear peak may not be seen, and there is also more danger of modal overlap with neighbouring modes, which complicates the analysis. So the approach is best for moderate damping. With loss factors around 1% the method overlaps with the transient decay method, and both can be used as a cross-check.

For a material with high damping no method based on resonance will work well. A forced vibration test is then the only option. Some kind of cyclic strain is imposed, at the chosen frequency. The energy dissipation per cycle, e.g. from the loop area in a stress-strain plot, give the damping value. The method relies on relatively elaborate and expensive equipment, and it is not easy to push to very high frequencies because of the difficulties of moving mass and possible resonances in the test machine. The method is less good for material with low damping because of additional damping coming from the grips or supports used to hold the test specimen.

(i) Tool steel has very low damping, so transient decay would be the best method. The material is robust and basically easy to test: a specimen would be made in the form of a rectangular beam, and resonant frequencies of bending modes measured. Support structures need to be carefully chosen to minimise additional damping. Non-contact sensing, e.g. by a microphone of laser vibrometer, would be a good choice.

(ii) Cork has rather high damping, and is rather fragile. It is also a natural material, so significant spatial variation might be expected. Probably the best method would be low-amplitude testing of a small specimen in a Dynamical Mechanical Analyser.

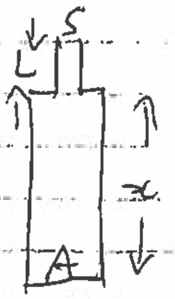
(iii) A ceramic honeycomb will be challenging to test. It would be expected to have low damping unless it contains cracks, and perhaps the detection of such cracks is one reason you might be trying to measure the damping. The thin walls of the structure are fragile and also very light in weight. It would be hard to hold a sample without causing damage, and the influence of surrounding air would need to be considered. A possible approach would involve exciting a specimen via a vibrating base, in a vacuum chamber, and monitoring the response from outside using a laser vibrometer.

(b)(i) In a Helmholtz resonator, the pressure is essentially uniform throughout the chamber, the only strong pressure gradient being around the neck where the pressure has to fall off to match the constant atmospheric pressure. The wavelength of sound at the resonant frequency plays no essential role in the mechanism. In an organ pipe, by contrast, the wavelength is crucial. Depending on the boundary conditions, the standing wave is determined by fitting 1/4 wave, 1/2 wave etc. into the length of the resonating tube.

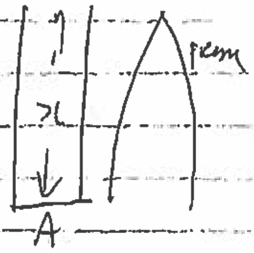
4 cont. (b)(ii) Volume $V = Ax$

So Helmholtz resonance frequency

$$\omega_H = c\sqrt{\frac{S}{VL}} = c\sqrt{\frac{S}{AxL}}$$



For an open/closed tube, the lowest standing wave resonance has ¼ wave in the tube: a pressure node at the open end, and an antinode at the closed end.



So wavelength $= 4x$

So resonance frequency $\omega_S = c \times \frac{2\pi}{4x} = \frac{\pi c}{2x}$

So ratio $\frac{\omega_H}{\omega_S} = \frac{2}{\pi}\sqrt{\frac{S}{A}\cdot\frac{x}{L}}$

For the Helmholtz resonator approximation to work, we want this ratio to be $\ll 1$, to justify ignoring wavelength effects. So want S small compared to A, and effective length L not too small compared to x.

(b)(iii) As the neck is opened out progressively, the lowest mode of the air must gradually "morph" from a Helmholtz resonance into a standing wave. The frequency will rise as S increases, until the wavelength begins to be significant compared to the length x, until a standing wave condition is reached

4C6 examiners comments:

Q1 Experimental modal analysis: 30 Attempts, Mean 14.7/20.
Generally well done. No-one scored full marks because I insisted on getting the right sign of modal amplitude. Many did, but they made other mistakes. So 19/20 was a popular mark..

Q2 Tuned vibration absorber: 24 Attempts, Mean 13.1/20
Quite well done. Many had trouble simplifying expressions for small λ but I was generous and allowed for unsimplified expressions. Part (c) asked for a sketch. Why so few sketches?!

Q3 Modes of a drum: 27 Attempts, Mean 13.7/20
Only one candidate properly identified the repeated modes. I had to mark generously on account of this. Generally well done, but good answers for Rayleigh's method had to say if the frequencies were going up or down. Interlacing wasn't enough.

Q4 Measurement of damping/Helmholtz resonator: 21 Attempts, Mean 15.5/20
Good answers to part (a) on measurement of damping. Helmholtz was OK, but not much critique on the assumptions made.