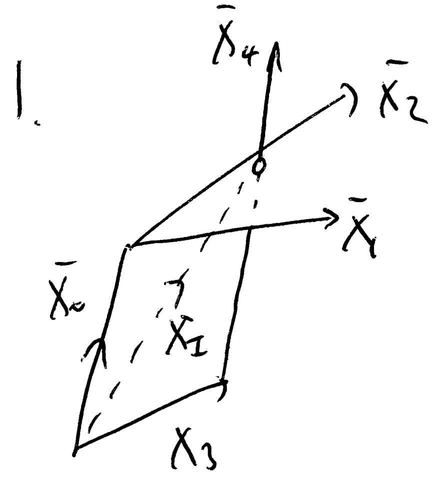
IA MATHEMATICS, 1997 Paper 4

1.



$$\bar{X}_I = \bar{X}_3 + \alpha_4 \bar{X}_4$$
$$= \hat{X}_0 + \alpha_1 \bar{X}_1 + \alpha_2 \bar{X}_2$$

$$\therefore \begin{pmatrix}0\\3\\1\end{pmatrix} + \alpha_1\begin{pmatrix}2\\-4\\-4/3\end{pmatrix} + \alpha_2\begin{pmatrix}3\\3\\1\end{pmatrix} = \begin{pmatrix}2\\4\\1\end{pmatrix} + \alpha_4\begin{pmatrix}0\\-8\\4\end{pmatrix}$$

$\therefore\ 2\alpha_1 + 3\alpha_2 = 2$

$-4\alpha_1 + 3\alpha_2 + 8\alpha_4 = 1 \longrightarrow 9\alpha_2 + 8\alpha_4 = 5$

$\frac{4}{3}\alpha_1 + \alpha_2 - 4\alpha_4 = 0 \longrightarrow \alpha_2 + 4\alpha_4 = 4/3$

$\therefore\ \alpha_4 = 1/4,\ \alpha_2 = 1/3,\ \alpha_1 = 1/2$

$\therefore\ \underline{X_I = (2\ 2\ 2)}$

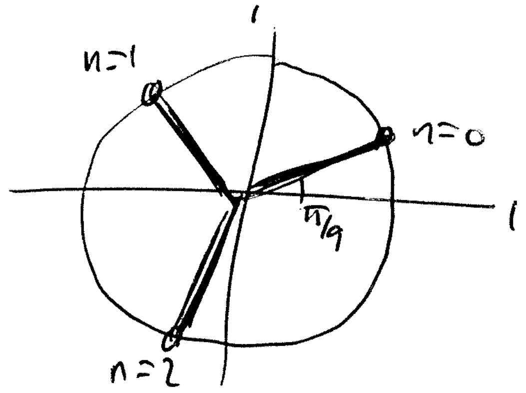
2. (a) $z^3 = \frac{1}{\sqrt{3}} + i$ $\qquad z = x + iy = re^{i\theta}$

$$r = \sqrt{x^2 + y^2} = 2/\sqrt{3}$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\sqrt{3} = \frac{\pi}{3} + 2n\pi$$

$$\therefore z = \left(\frac{2}{\sqrt{3}}\right)^{1/3} e^{i\left(\frac{\pi}{9} + n\frac{2}{3}\pi\right)}$$

3 roots as shown



(b) $$\frac{x - \log(1+x)}{e^x \sin x - x\cos x} = \frac{x - \left(x - \frac{x^2}{2!} + \cdots\right)}{\left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right) - x\left(1 - \frac{x^2}{2!} + \cdots\right)}$$

$$= \frac{x^2/2}{\cancel{x} - x^3/6 + x^2 \cdots \cancel{-x} + x^3/2} = \frac{x^2/2}{x^2}$$

$$\therefore \lim_{x \to 0} = \frac{1}{2}$$

3. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 24e^{-x}\sin 2x$; $y=0$ & $\frac{dy}{dx}=0$ @ $x=0$.

CF: $\lambda^2+2\lambda+2=0$ $\lambda = \frac{-2\pm\sqrt{\cancel{4-8}}}{2}$ -4 $= -1\pm i$

$\Rightarrow$ $y = e^{-x}(A\cos x + B\sin x)$

PI: try $y = e^{-x}(C\cos 2x + D\sin 2x)$

$$\frac{dy}{dx} = -Ce^{-x}\cos 2x - De^{-x}\sin 2x - 2Ce^{-x}\sin 2x + 2De^{-x}\cos 2x$$

$$\frac{d^2y}{dx^2} = Ce^{-x}\cos 2x + 2Ce^{-x}\sin 2x + De^{-x}\sin 2x - 2De^{-x}\cos 2x$$
$$+2Ce^{-x}\sin 2x - 4Ce^{-x}\cos 2x - 2De^{-x}\cos 2x - 4De^{-x}\sin 2x$$

gather: $25e^{-x}\sin 2x = e^{-x}\cos 2x(C-2D-4C-2D) = e^{-x}\cos 2x(-3C-4D)$
$+ e^{-x}\sin 2x(2C+D+2C-4D) + e^{-x}\sin 2x(4C-3D)$
$+ e^{-x}\cos 2x(-2C+4D)$
$+ e^{-x}\sin 2x(-2D-4C)$ $+ e^{-x}\cos 2x(4D)$
$+ e^{-x}\cos 2x(2C)$ $+ e^{-x}\sin 2x(-4C)$
$+ e^{-x}\sin 2x(2D)$

$\therefore$ $-3C \cancel{-4D} \cancel{+4D} = 0$ $\underline{C=0}$
$\cancel{4C} - 3D \cancel{-4C} = 24$ $\underline{D=-8}$

$\therefore$ sol^n: $\underline{y = e^{-x}(A\cos x + B\sin x - 8\sin 2x)}$

BC's: $y=0$ @ $x=0$ $\Rightarrow$ $A=0$
$\frac{dy}{dx}=0$ @ $x=0$ $\Rightarrow$ $1\cdot(B-16)+(-1)(0)$ $\Rightarrow$ $B=16$

$\Rightarrow$ $\underline{y = 8e^{-x}(2\sin x - \sin 2x)}$

4. $c(u_{i+1} - u_{i-1}) = d(u_{i+1} - 2u_i + u_{i-1})$

let: $u_i = A\lambda^i$ $\therefore$ $A\lambda^i(\lambda(c-d) + 2d - (c+d)/\lambda) = 0$

$\therefore \lambda = \dfrac{-2d \pm \sqrt{[4d^2 + 4(c-d)(c+d)]}}{2(c-d)} = \dfrac{-d \pm c}{c-d} \Rightarrow \lambda = 1, \dfrac{d+c}{d-c}$

$\therefore$ general solution is $\underline{u_i = A + B\left(\dfrac{d+c}{d-c}\right)^i}$

$c, d > 0$. If $c > d$ then $(d+c)/(d-c) < 0$ so solution is oscillating depending on i = odd/even

If $u_i = 0$ @ $i = 1$ $\quad A = -B(d+c)/(d-c)$

$u_i = 1$ @ $i = I$ $\quad 1 = B\left(\dfrac{d+c}{d-c}\right)^I - \overbrace{\left(\dfrac{d+c}{d-c}\right)B}^{A}$

$= B\left(\dfrac{d+c}{d-c}\right)\left(\left(\dfrac{d+c}{d-c}\right)^{I-1} - 1\right)$

$$\therefore u_i = \frac{1 - \left(\frac{d+c}{d-c}\right)^{i-1}}{1 - \left(\frac{d+c}{d-c}\right)^{I-1}}$$

5. $A = QR$ $\quad \left[\underline{a}_1 \middle| \underline{a}_2 \middle| a_3\right] = \left[\bar{q}_1 \middle| \bar{q}_2 \middle| \bar{q}_3\right] \begin{bmatrix} 1 & r_{12} & r_{13} \\ & 1 & r_{23} \\ & & 1 \end{bmatrix}$

$\therefore \bar{a}_1 = \bar{q}_1 \, , \; \bar{a}_2 = \bar{q}_2 + r_{12}\bar{q}_1 \, , \; \bar{a}_3 = \bar{q}_3 + r_{23}\bar{q}_2 + r_{13}\bar{q}_1$

$$\Rightarrow \quad \bar{q}_j = \bar{a}_j - \sum_{i=1}^{j-1} r_{ij}\bar{q}_i \, .$$

$\bar{q}_i \cdot \bar{q}_j = 0 \;\; \forall \;\; i \neq j$

$$\Rightarrow \quad \bar{a}_2 \cdot \bar{q}_1 = \underbrace{\bar{q}_2 \cdot \bar{q}_1}_{\to 0} + r_{12}\bar{q}_1 \cdot \bar{q}_1$$

$$\therefore \quad r_{12} = \bar{a}_2 \cdot \bar{q}_1 / \bar{q}_1 \cdot \bar{q}_1$$

Similarly : $\quad r_{13} = \bar{a}_3 \cdot \bar{q}_1 / \bar{q}_1 \cdot \bar{q}_1$

& : $\quad r_{23} = \bar{a}_3 \cdot \bar{q}_2 / \bar{q}_2 \cdot \bar{q}_2 \, .$

6(a). $p(\text{First}) = \dfrac{n}{n+m}$

$p(\text{last}) = \dfrac{n-1}{n+m-1}$

$p(\text{First and last}) = \underline{\underline{\left(\dfrac{n}{n+m}\right)\left(\dfrac{n-1}{n+m-1}\right)}}$

(b) There are $\dfrac{(n+m)!}{n!\,m!}$ possible permutations of A and B

Of these, there are n permutations for which all B are continuous

So the probability $= \underline{\underline{n \dfrac{n!\,m!}{(n+m)!}}}$

(c) This is only possible if $m \geq n-1$

If $m = n-1$, probability $= \dfrac{n!\,m!}{(n+m)!}$

If $m = n$, probability $= 2 \dfrac{n!\,m!}{(n+m)!}$

If $m \geq n$ We have n lots of A then B

and $(m-n)$ lots of B alone

Number of permutations [A then B]'s and [B]'s is

$\binom{m}{m-n} = \dfrac{m!}{(m-n)!\,n!}$

Total number of permutation of A & B is

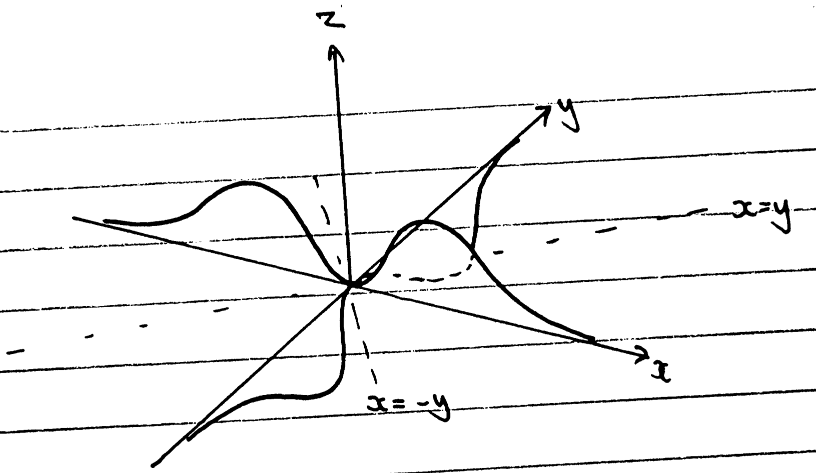
$\binom{m+n}{n} = \dfrac{(m+n)!}{m!\,n!}$

Probability of no consecutive A's is one over other :

prob $= \dfrac{m!}{(m-n)!\,n!} \dfrac{m!\,n!}{(m+n)!}$

$= \underline{\underline{\dfrac{(m!)^2}{(m-n)!\,(m+n)!}}}$

7.



Using $\frac{dz}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ where $z = u.v$:

$$\frac{\partial z}{\partial x} = (y^2 - x^2 + 1) \,.\, 2x \,.\, e^{-(x^2+y^2)}$$

$$\frac{\partial z}{\partial y} = (y^2 - x^2 - 1) \,.\, 2y \,.\, e^{-(x^2+y^2)}$$

z is stationary when $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$, ie when:

$x = 0$ or $x^2 - y^2 = 1$

and $y = 0$ or $y^2 - x^2 = 1$

So either $x = 0$ and $y = 0$ or ± 1

or $y = 0$ and $x = \pm 1$ [can't have $x^2 - y^2 = y^2 - x^2 = 1$]

By inspection of sketch, $(0,0)$ is saddle, $(1,0)$ and $(-1,0)$ are maxima, and $(0,1)$ and $(0,-1)$ are minima.

Direction of steepest ascent is $\underline{\nabla} z$, and at $x=y=1$, $\underline{\nabla} z = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

8.(a) Step response $= \int(\text{impulse response})\, dt$ + constant

$= -2e^{-t/2}$ + constant $= 0$ at $t = 0$

$= 2 - 2e^{-t/2}$

(b)

$$y(t) = \int_{-\infty}^{t} f(\tau)\, e^{-\frac{1}{2}(t-\tau)}\, d\tau$$

$$= e^{-\frac{1}{2}t} \int_0^t \tau^2 e^{-\tau} e^{\frac{\tau}{2}}\, d\tau$$

$$\int \tau^2 e^{-\frac{\tau}{2}}\, d\tau = \left[-2\tau^2 e^{-\frac{\tau}{2}}\right] + 2\int 2\tau e^{-\frac{\tau}{2}}\, d\tau = \left[-2\tau^2 e^{-\frac{\tau}{2}}\right] + 2\left[-4\tau e^{-\frac{\tau}{2}}\right] + 2\int -4\, e^{-\frac{\tau}{2}}$$

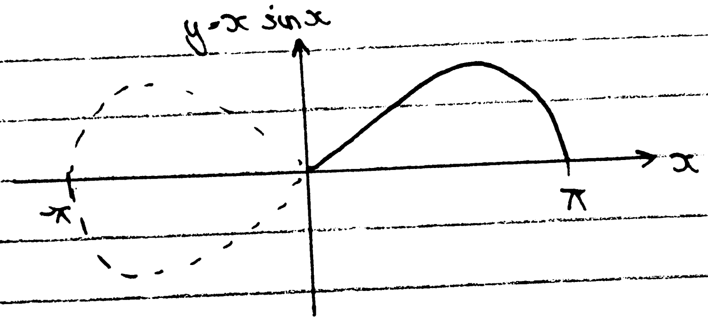
$$= \left[(-2\tau^2 - 8\tau - 16)\, e^{-\frac{\tau}{2}}\right]$$

$$y(t) = e^{-\frac{1}{2}t} \left[(-2\tau^2 - 8\tau - 16)\, e^{-\frac{\tau}{2}}\right]_0^t$$

$$= e^{-\frac{1}{2}t} \left[(-2t^2 - 8t - 16)\, e^{-\frac{t}{2}} + 16\right]$$

$$= -(2t^2 + 8t + 16)\, e^{-t} + 16 e^{-t/2}$$

9(a)



y is continuous for both odd and even functions but $\frac{dy}{dx}$ is discontinuous for the even function so the odd function converges more rapidly.

$$b_n = \frac{2}{\pi}\int_0^\pi x \sin x \ \sin nx \ dx$$

$$\pi b_n = \int_0^\pi x\left[\cos(n-1)x - \cos(n+1)x\right] dx$$

$$= \left[\overset{=0 \text{ for all } n}{\cancel{x\,\frac{\sin(n-1)x}{n-1}}}\right]_0^\pi - \int_0^\pi \frac{\sin(n-1)x}{n-1} - \left[\overset{=0 \text{ for all } n}{\cancel{x\,\frac{\sin(n+1)x}{n+1}}}\right]_0^\pi + \int_0^\pi \frac{\sin(n+1)x}{n+1}\,d$$

$$= \left[\frac{\cos(n-1)x}{(n-1)^2}\right]_0^\pi - \left[\frac{\cos(n+1)x}{(n+1)^2}\right]_0^\pi$$

$$= 0 \quad \text{if } n \text{ is odd}$$

If n is even, $\pi b_n = \frac{-2}{(n-1)^2} - \frac{-2}{(n+1)^2}$

$$= \frac{2(n+1)^2 - 2(n-1)^2}{(n^2-1)^2}$$

$$b_n = \frac{8n}{\pi(n^2-1)^2}$$

10 (a) $s^2\bar{y}(s) - 4\bar{y}(s) = \dfrac{1}{s^2-1}$

$$\bar{y}(s) = \frac{1}{(s+2)(s-2)(s+1)(s-1)}$$

$$= \frac{-1/12}{s+2} + \frac{1/12}{s-2} + \frac{1/6}{s+1} + \frac{-1/6}{s-1} \quad \text{by cover-up rule}$$

so $y(t) = \tfrac{1}{12}(e^{2t} - e^{-2t}) + \tfrac{1}{6}(e^{-t} - e^{t})$

$$= \underline{\underline{\tfrac{1}{6}\sinh 2t - \tfrac{1}{3}\sinh t}}$$

Check $\dot{y}(t) = \tfrac{1}{3}\cosh 2t - \tfrac{1}{3}\cosh t$

$$\ddot{y}(t) = \tfrac{2}{3}\sinh 2t - \tfrac{1}{3}\sinh t$$

So $y(0) = \dot{y}(0) = 0$ and $\ddot{y} - 4y = \sinh t$

(b) $s^2\bar{y}(s) - 1 + 2s\bar{y}(s) = -\dfrac{1}{(s+1)^2+1}$

$$(s^2+2s)\bar{y}(s) = 1 - \frac{1}{(s+1)^2+1}$$

$$\bar{y}(s) = \frac{(s+1)^2}{s(s+2)(s^2+2s+2)}$$

$$= \frac{1/4}{s} + \frac{-1/4}{s+2} + \frac{Cs+D}{s^2+2s+2} \quad \text{by cover-up rule}$$

Let $s=-1$: $\dfrac{(s+1)^2}{s(s+2)(s^2+2s+2)} = 0 = -\dfrac{1}{4} - \dfrac{1}{4} + \dfrac{D-C}{1}$

$$D - C = \tfrac{1}{2}$$

Let $s=1$: $\dfrac{4}{15} = \dfrac{1}{4} - \dfrac{1}{12} + \dfrac{C+D}{5}$

$$C + D = \frac{-15}{12} + \frac{5}{12} + \frac{16}{12} = \frac{1}{2}$$

So $D = \tfrac{1}{2}$ and $C = 0$

$$\bar{y}(s) = \frac{1/4}{s} - \frac{1/4}{s+2} + \frac{1/2}{(s+1)^2+1}$$

$$y(t) = \underline{\underline{\tfrac{1}{4} - \tfrac{1}{4}e^{-2t} + \tfrac{1}{2}e^{-t}\sin t}}$$

Check: $\dot{y}(t) = \tfrac{1}{2}e^{-2t} - \tfrac{1}{2}e^{-t}\sin t + \tfrac{1}{2}e^{-t}\cos t$

$$\ddot{y}(t) = -e^{-2t} + \cancel{\tfrac{1}{2}e^{-t}\sin t} - \tfrac{1}{2}e^{-t}\cos t - \tfrac{1}{2}e^{-t}\cos t - \cancel{\tfrac{1}{2}e^{-t}\sin t}$$

$y(0) = 0$, $\dot{y}(0) = 1$ and $\ddot{y} + 2\dot{y} = -e^{-t}\sin t$