ENGINEERING TRIPOS PART IA

Monday 9 June 1997 9 to 12

Paper 4

MATHEMATICAL METHODS

Answer not more than **eight** *questions of which not more than* **four** *may be taken from Section A and not more than* **four** *from Section B.*

The **approximate** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

(TURN OVER

SECTION A

Answer not more than **four** *questions from this section.*

1 The intersection of a line $\underline{X}_4$ with a plane defined by $\underline{X}_1$ and $\underline{X}_2$ is sketched in Fig. 1. If the line intersects the plane at the point defined by $\underline{X}_p = \underline{X}_3 + \alpha\,\underline{X}_4$, write down a similar expression for $\underline{X}_p$ in terms of the vectors $\underline{X}_0$, $\underline{X}_1$ and $\underline{X}_2$. [8]

Hence, if $\underline{X}_0 = (0\ \ 3\ \ 1)^t$, $\underline{X}_1 = (2\ \ -4\ \ \frac{4}{3})^t$, $\underline{X}_2 = (3\ \ 3\ \ 1)^t$, $\underline{X}_3 = (2\ \ 4\ \ 1)^t$, $\underline{X}_4 = (0\ \ -8\ \ 4)^t$ find the co-ordinates of the intersection. [12]

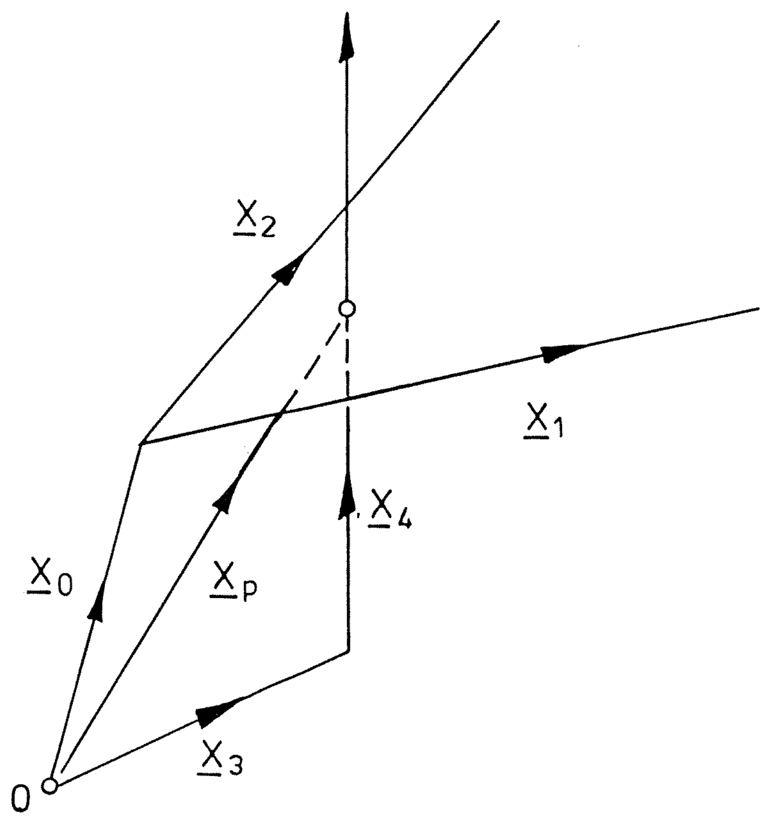

Fig. 1

2 (a) Plot on an Argand diagram all the solutions of $z^3 = \frac{1}{\sqrt{3}} + i$. [10]

(b) Find

$$\lim_{x \to 0} \frac{x - \ln(1+x)}{e^x \sin x - x \cos x} \quad .$$ [10]

3 Find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 24e^{-x} \sin 2x \ .$$ [12]

Find the solution which satisfies the boundary conditions

$$y = 0, \ \frac{dy}{dx} = 0 \ \text{ at } \ x = 0 \ .$$ [8]

4 One-dimensional fluid flow can be modelled by the difference equation below which relates a sequence of velocities u_i

$$c(u_{i+1} - u_{i-1}) = d(u_{i+1} - 2u_i + u_{i-1})$$

for values of i = 1,2,..., where c and d are positive constants.

Find the solution to this difference equation for a case with $u_i = 0$ at $i = 1$ and $u_i = 1$ at $i = I$. [16]

Hence show that the solution will be oscillatory if $c > d$. [4]

5 It is desired to write a 3×3 matrix A as the product of matrices Q and R where R has the structure

$$R = \begin{bmatrix} 1 & r_{12} & r_{13} \\ 0 & 1 & r_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

If A and Q are written in terms of their column vectors

$$A = \left[\begin{array}{c|c|c} & & \\ \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \\ & & \end{array} \right] \text{ and } Q = \left[\begin{array}{c|c|c} & & \\ \underline{q}_1 & \underline{q}_2 & \underline{q}_3 \\ & & \end{array} \right] ,$$

show that the vectors $\underline{q}_j$ can be found from the recurrence

$$\underline{q}_j = \underline{a}_j - \sum_{i=1}^{j-1} r_{ij} \underline{q}_i \ .$$ [10]

Find the values of r_{ij} given that the column vectors of Q are orthogonal, i.e. $\underline{q}_i . \underline{q}_j = 0$ if $i \neq j$. [10]

(TURN OVER

SECTION B

Answer not more than **four** *questions from this section.*

6 A machine is used to manufacture custom spoilers for two types of sports car (Car A and Car B). Each day, in a random order, n are produced for Car A and m for Car B.

(a) Find the probability that the first and last one produced on a given day are both for Car A. [6]

(b) What is the probability that the m spoilers for Car B are produced consecutively? [6]

(c) By considering the intervals between spoilers produced for Car B, find the probability that no two spoilers for Car A are produced consecutively, distinguishing between cases for which this is and is not possible. [8]

7 A function is defined by the equation

$$z = (x^2 - y^2)\, e^{-(x^2+y^2)}$$

Sketch the function, and find and classify its stationary points. [12]

Find the unit vector pointing in the direction of steepest ascent for z at $x = y = 1$. [8]

8 (a) If the impulse response of a stationary linear system is $y = e^{-t/2}$ for $t > 0$, find the step response. [10]

(b) Find the response of the same system if the input is $x = t^2 e^{-t}$ for $t > 0$ and $y = 0$ for $t \leq 0$. You may wish to use the convolution integral: [10]

$$f(t) = \int_{-\infty}^{t} f(\tau)\, \delta(t-\tau)\, d\tau \quad .$$

9 (a) Sketch the function

$$y = x \sin x$$ [10]

over the range $0 \leq x \leq \pi$.

(b) Outside this range, $y(x)$ is to be extended to make a function with period 2π so that it is *either* odd *or* even. Identify which of these will have the more rapidly converging Fourier series, and calculate the coefficients of the first three terms of this series. [10]

10 Solve the following equations using Laplace Transforms:

(a) $\ddot{y} - 4y = \sinh t,$ $\quad y(0) = \dot{y}(0) = 0$ [10]

(b) $\ddot{y} + 2\dot{y} = -e^{-t} \sin t,$ $\quad y(0) = 0,\ \dot{y}(0) = 1$. [10]

END OF PAPER