ENGINEERING TRIPOS PART IB

Friday 9 June 2006 9 to 11

Paper 7

MATHEMATICAL METHODS

Answer not more than ***four*** *questions.*

Answer not more than ***two*** *questions from each section.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version: archive

SECTION A

*Answer not more than **two** questions from this section*

1 (a) Prove Gauss' theorem, given the coordinate-free definition of divergence:

$$\nabla \cdot \mathbf{F} = \lim_{\delta V \to 0} \frac{1}{\delta V} \oint_S \mathbf{F} \cdot \mathbf{dA}$$

where S is the surface that encloses a small volume δV. [6]

(b) A device to harness solar energy consists of a large polythene chimney joined to the ground by a thin net. The air within the chimney expands and passes through a turbine at the top of the chimney.

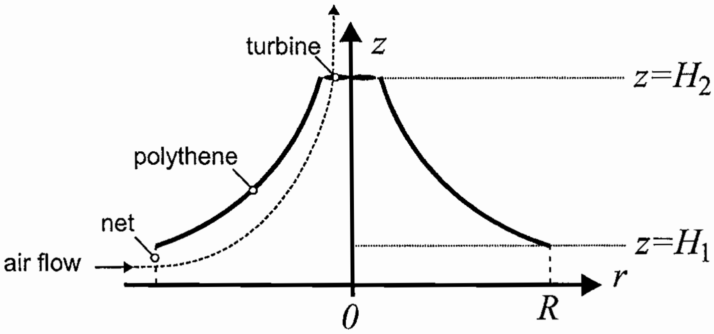

Figure 1

The device is rotationally-symmetric about the z-axis. Its radius is given by $r(z) = R$ from $z = 0$ to $z = H_1$ and $r(z) = RH_1/z$ from $z = H_1$ to $z = H_2$. What is the volume of air within the device? [4]

(c) The velocity vector field is denoted $\mathbf{u}$. The velocity through the net is $\mathbf{u} = -U_1\hat{\mathbf{e}}_\mathbf{r}$ and the velocity through the turbine is $\mathbf{u} = U_2\hat{\mathbf{e}}_\mathbf{z}$, where $\hat{\mathbf{e}}_\mathbf{r}$ and $\hat{\mathbf{e}}_\mathbf{z}$ are unit vectors in the radial and axial directions respectively and U_1 and U_2 are constants. Assuming that $\nabla \cdot \mathbf{u}$ is constant within the device, evaluate $\nabla \cdot \mathbf{u}$ in terms of R, H_1, H_2, U_1 and U_2. [6]

(d) Evaluate the flux of $\nabla \times \mathbf{u}$ through the cross-section of the device at $z = H_1/2$. This cross section is circular and has radius R. [4]

2 The vector field $\mathbf{B}$ is expressed in terms of the Cartesian co-ordinate system (x, y, z) and the associated unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ as

$$\mathbf{B} = yz\frac{df}{dx}\ \mathbf{i} + fz\ \mathbf{j} + fy\ \mathbf{k}\ ,$$

where $f(x)$ is a function only of x.

(a) Show that $\mathbf{B}$ is a conservative vector field and find its scalar potential ϕ. [4]

(b) Evaluate the line integral $\int \mathbf{B} \cdot d\mathbf{l}$ from (-1,1,1) to (1,1,1) when:

(i) $f(x) = \sin(\pi x/2)$;
(ii) $f(x) = \cos(\pi x/2)$. [4]

(c) The surface S encloses the volume V that is a sphere of radius 1 centred at $(x, y, z) = (0, 1, 1)$. If the net flux of $\mathbf{B}$ through S is equal to zero and $f(x)$ is differentiable within S, what condition or conditions could $f(x)$ satisfy? [6]

(d) A new vector field is defined as $\mathbf{C} = \psi\mathbf{B}$. If $\mathbf{C}$ is a conservative vector field and $\mathbf{B} = \nabla\phi$, find an expression relating $\nabla\psi$ to $\nabla\phi$. Show that this expression is satisfied by $\psi = g(\phi)$, where g is any differentiable function of ϕ. [6]

3 A uniform beam has length L and is aligned along the x axis, as shown in figure 2.

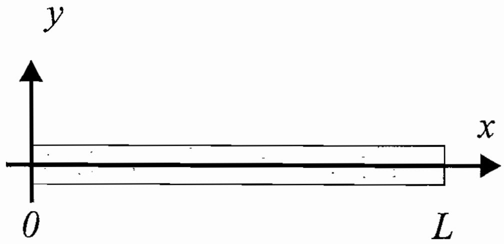

Figure 2

Small undamped transverse vibrations of this beam are governed by the following partial differential equation for the deflection $y(x,t)$:

$$c^2 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0$$

where c is a constant.

(a) We wish to find a solution to this equation by the separation of variables: $y(x,t) = X(x)T(t)$. Find the governing ordinary differential equations for X and T. Confirm that the following expressions for X and T satisfy these equations and find the relationship between ω, k and c. [10]

$$\begin{aligned} X(x) &= A\cos(kx) + B\sin(kx) + C\cosh(kx) + D\sinh(kx) \\ T(t) &= P\cos(\omega t) + Q\sin(\omega t) \end{aligned}$$

(b) When the beam is simply supported at both ends the boundary conditions at $x = 0$ and $x = L$ are:

$$y = 0, \qquad \frac{\partial^2 y}{\partial x^2} = 0 \qquad \text{for all } t.$$

Find all solutions to the above governing partial differential equation, subject to these boundary conditions and with zero initial velocity: i.e. $dy/dt = 0$ for all x at $t = 0$. [10]

SECTION B

Answer not more than ***two*** *questions from this section*

4 The 3×3 matrix A is given by

$$A = \begin{bmatrix} 1 & a & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) Find a value of a which produces *real* eigenvalues and *orthogonal* eigenvectors. [2]

(b) Find a value of a which makes A *singular*. For this value of a, find the eigenvalues and the eigenvector corresponding to the largest eigenvalue. [4]

(c) Find the two values of a for which the matrix A has only 2 *distinct* eigenvalues. For one of these values of a, find the eigenvalues of A and the corresponding set of eigenvectors. Explain why the resulting A is *not* diagonalizable. [6]

(d) For a matrix A, outline the *power method* which can be used to estimate the eigenvector corresponding to the largest eigenvalue of A from some initial guess. For a taking the value found in Part (b), use this method to estimate the eigenvector corresponding to the largest eigenvalue of A using 4 iterations and an initial guess of

$$\mathbf{u}_0 = [0, 0, 1]^T$$

By comparing your result with the true unit eigenvector, comment on the accuracy of the estimate. What is the *convergence factor* in this case? [8]

5 (a) By performing LU decomposition, or otherwise, on the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ -2 & 2 & 1 & -4 \\ 2 & -2 & -1 & 4 \end{bmatrix}$$

find a basis for each of the fundamental subspaces of A. [12]

(b) The joint probability density function of two continuous random variables, X and Y, is given by

$$p(x,y) = \begin{cases} \alpha(xy+2), & 0 \leq x \leq 2, \;\; 0 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the value of α.

(ii) Find $E[X]$, $E[XY]$ and $E[X^2]$.

(iii) Explain why, for this problem, $E[X] = E[Y]$ but $E[XY] \neq E[Y]^2$. [8]

6 (a) The Moment Generating Function (MGF) for the binomial distribution with parameters n and p is

$$g(z) = (q + pz)^n$$

where $q = 1 - p$. By substituting $\lambda = np$ in the above, show that this MGF becomes

$$g(z) = \left[1 + \frac{\lambda}{n}(z-1)\right]^n$$

Taking the limit as $n \to \infty$, show that $g(z)$ tends to the MGF of a Poisson distribution with parameter λ. [7]

(b) 10% of the chocolates produced in a factory turn out to be deformed. Find the probability that in a sample of 10 chocolates chosen at random, more than 2 will be deformed, by

(i) using the binomial distribution;

(ii) using the Poisson approximation to the binomial distribution. [6]

(c) Repeat part (b) in the case that 20% of the chocolates are deformed and the sample size changes to 5. [3]

(d) By considering the mean and variance of both the binomial and the Poisson distributions, state, giving reasons, the conditions under which the Poisson distribution is a good approximation to the binomial distribution. Using this, comment on your results for (b) and (c). [4]

END OF PAPER