ENGINEERING TRIPOS PART IIA

Thursday 1 May 2003 9.00 to 10.30

Module 3F2

SYSTEMS AND CONTROL

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1 A linear system is given in state-space form as:

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + [B_1 \;\; B_2]\begin{bmatrix} u \\ d \end{bmatrix} \\ \underline{y} &= C\underline{x} + Dd \end{aligned}$$

where u is a scalar control input, d is a scalar disturbance, $\underline{x}$ is the state vector and $\underline{y}$ is the output vector.

(a) Show that the transfer function matrix from u to $\underline{y}$ is $C(sI - A)^{-1}B_1$, and find the transfer function matrix from d to $\underline{y}$. [25%]

(b) State the definition of the *matrix exponential* function e^M, where M is a square matrix. Verify that $\underline{x}(t) = e^{At}\underline{x}(0)$ satisfies the equation $\dot{\underline{x}}(t) = A\underline{x}(t)$. If $u(t) = 0$, write down the expression for the state $\underline{x}(t)$ which results from an initial condition $\underline{x}(0)$ and disturbance signal $d(t)$. [25%]

(c) A linearised model of a small aircraft in a particular cruising condition has:

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & -2 & 0 \\ -100 & 100 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -100 & 100 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If $\underline{x}(0) = 0$, $u(t) \equiv 0$, and $d(t)$ is a unit step, show that $\underline{y}(t) \approx \begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix}$ for small enough t. [30%]

(d) Consider the model in (c). If $B_1 = \begin{bmatrix} -1 \\ 0 \\ -20 \\ 0 \end{bmatrix}$ and proportional feedback from the first output is used (namely $u(t) = ky_1(t)$), show that the closed loop is not asymptotically stable for any choice of the gain k. (*Hint:* You are not required to calculate the position of all the closed loop poles.) [20%]

2 The figure shows the block-diagram of a radial positioning system for a read/write head in a hard-disk assembly. y is the radial position, and r is its required value. u is the force applied by the position controller. m is the mass of the head, and λ is a viscous damping coefficient.

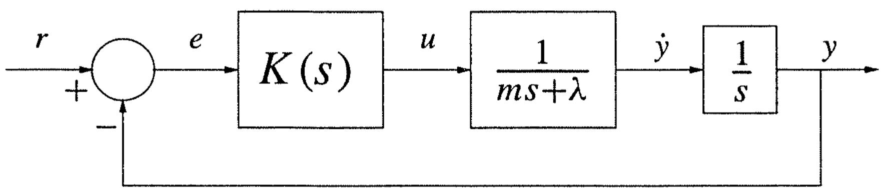
(a) It is proposed to use only proportional gain, namely $K(s) = k$, a constant. Draw the root-locus diagram for $k > 0$. Hence (or otherwise) find the range of values of k which will give a closed-loop damping factor no smaller than $1/\sqrt{2}$. When tracking a ramp $r(t) = \alpha t$, what is the smallest steady-state error achievable if k is confined to this range? [40%]

(b) It is proposed to use a phase advance compensator of the form

$$K(s) = \frac{k(s + \frac{\lambda_0}{m})}{s + p}.$$

Assuming that $\lambda = \lambda_0$, find the smallest value of p which will allow the steady-state error to be reduced to $m\alpha/10\lambda_0$ or less, with a closed loop damping factor of $1/\sqrt{2}$, when the ramp defined in (a) is being tracked. [40%]

(c) The compensator defined in (b) is used, but the damping in the read-write head varies, so that the actual value of λ varies from its nominal value λ_0. Explain, briefly, how the root-locus method can be used to investigate the variation of the closed-loop poles as λ varies, when the compensator transfer function $K(s)$ remains fixed. [20%]



3 The water temperature θ in a heated swimming pool is related to the heater power u, and the ambient air temperature, θ_a, by the equation

$$T\frac{d\theta}{dt} = ku - (\theta - \theta_a)$$

where time t is measured in hours, $T = 1$ hour, and k is a constant. The ambient temperature θ_a is assumed to vary sinusoidally with a period of 24 hours:

$$\theta_a(t) = M\sin(\omega t + \phi) + N$$

where $\omega = 2\pi/24$ and N is the mean ambient temperature, which is assumed to be constant. The parameters M, N and ϕ are unknown.

(a) Choosing state variables $x_1 = \theta$, $x_2 = \theta_a$, $x_3 = \dot{\theta}_a$, $x_4 = N$, and state vector $\underline{x} = [x_1, x_2, x_3, x_4]^T$, show that the following state-space equation holds:

$$\dot{\underline{x}} = \begin{bmatrix} -\frac{1}{T} & \frac{1}{T} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\omega^2 & 0 & \omega^2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} k/T \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

[30%]

(b) Show that the state vector $\underline{x}$ defined in (a) is observable from measurements of the water temperature alone. [30%]

(c) Explain what is meant by a *state observer*. Describe the structure of a state observer for the system defined in (a) if both the water temperature θ and the air temperature θ_a are measured. (You are *not* required to compute suitable observer gains.) [20%]

(d) If only the water temperature θ is measured, how many entries does the observer gain matrix have? Determine two of these entries if all the observer eigenvalues are to be placed at $-1\ \mathrm{h}^{-1}$. (*Hint*: The sum of the eigenvalues depends on only one of these entries, and the product depends on only one other.) [20%]

4 The speed v of a car is governed by the equation (after scaling):

$$\frac{dv}{dt} = f - v^3$$

where f is the traction force produced by the engine. The force f depends on the throttle position u according to

$$\frac{df}{dt} = -2f + 2\tanh u$$

(a) Find the values of u and f required to establish equilibrium at speed $v = \frac{1}{2}$. Show that $v < 1$ in any equilibrium condition. [30%]

(b) Obtain a state-space model of the car, linearised at the equilibrium conditions found in (a) for $v = \frac{1}{2}$. [30%]

(c) Verify that the linear state-space model found in (b) is controllable from the throttle position u. Design a state-feedback controller for the speed ('cruise control') which results in the closed-loop poles being placed at -0.3 and -2. [40%]

END OF PAPER

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1. (a) $C(sI - A)^{-1}B_2 + D$
   (b) $x(t) = e^{At}x(0) + \int_0^t e^{A\tau}B_2 d(t-\tau)\, d\tau$
2. (a) $0 \leq k \leq \frac{\lambda^2}{2m}, \frac{2m\alpha}{\lambda}$.
   (b) $p \geq \frac{20\lambda}{m}$
3. (d) $l_1 = 4 - 1/T_1 = 3$, $l_4 = T/\omega^2 = 14.59$
4. (a) $f_e = 1/8$, $u_e = \tanh^{-1}(1/8) = 0.1257$
   (b) $A = \begin{bmatrix} -3/4 & 1 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \frac{126}{64} \end{bmatrix}$ (for $x_1 = v$, $x_2 = f$)
   (c) $k_1 = -0.2857$ $k_2 = -0.2286$