Worked solutions for 4B23 Optical Fibre Communication 2025/26 paper version 3

Question 1

a)

The fibre has low loss which in turn reduces the amount of noise due to ASE from optical amplifiers present, but also the high dispersion and large effective area reduce the nonlinear impairments allowing the SNR of the link to be increased.

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b)

$$D = 22 \text{ ps/nm/km}$$

At $\lambda = 1550$ nm the frequency 100 GHz=0.8 nm so 200 GBd occupies 1.60 nm

Hence the delay from 11,000 km is

$$11{,}000 \times 22 \times 1.6 = 387.2 \text{ ns}$$

Sampling rate is 225 GSa/s

Therefore number of samples corresponding to 387.2 ns is

$$N_{CD} = 387.2 \times 10^{-9} \times 225 \times 10^{9} = 87{,}120 \text{ taps}$$

This is a reasonable approximation however the filter needs to work over the whole of the C-band which from lectures goes from 1530-1565 nm so we should consider what happens at the edge of the C-band.

At 1565 nm the number of taps will increase by $\left(\frac{1565}{1550}\right)^2$ i.e. increases to 88,815 taps (rounding up), so this is an estimate of the minimum number of taps for any C-band WDM channel.

[20]

c)

We note

$$\log_2(N_{CD}) = 16.4$$

So start with

$$N = 2^{18}$$

$$N_{cm} = \frac{2^{18} \times (18 + 1)}{2^{18} - 88815 + 1} = 28.74$$

$$N_{cm} = \frac{2^{19} \times (19 + 1)}{2^{19} - 88815 + 1} = 24.08$$

$$N_{cm} = \frac{2^{20} \times (20 + 1)}{2^{20} - 88815 + 1} = 22.94$$

$$N_{cm} = \frac{2^{20} \times (20 + 1)}{2^{20} - 88815 + 1} = 22.97$$

$$N_{cm} = \frac{2^{21} \times (21 + 1)}{2^{21} - 88815 + 1} = 23.50$$

So minimum is 22.94 hence

Power is

$$P = 2 \times 22.9 \times 225 \times 10^9 \times 0.3 \times 10^{-12} = 3.1W$$

[30]

d) The PMD coefficient is 0.2 ps km$^{-1/2}$ and hence over 11,000 km the mean DGD is $0.2 \times \sqrt{11{,}000}$=21 ps
From formula sheet

$$P(\Delta\tau > x) \approx \frac{4}{\pi}\frac{x}{\langle\Delta\tau\rangle}\exp\left(-\frac{4x^2}{\pi\langle\Delta\tau\rangle^2}\right)$$

Hence

$$-12 = \log_{10}\left[\frac{4}{\pi}\frac{x}{\langle\Delta\tau\rangle}\exp\left(-\frac{4x^2}{\pi\langle\Delta\tau\rangle^2}\right)\right]$$

Therefore noting $\log_a b = \ln(b)/\ln(a)$

$$-12\ln(10) = \ln\left[\frac{4}{\pi}\frac{x}{\langle\Delta\tau\rangle}\exp\left(-\frac{4x^2}{\pi\langle\Delta\tau\rangle^2}\right)\right] \approx -\frac{4x^2}{\pi\langle\Delta\tau\rangle^2}$$

Two approaches, first is the approximation giving

$$\frac{x}{\langle\Delta\tau\rangle} = \sqrt{\frac{12\pi\ln(10)}{4}} = 4.66$$

Hence

$$x = 97.7 \text{ ps}$$

If we assume sampling rate of 225 GSa/s in the DSP

$$N_{tap} = 1 + 225 \times 10^9 \times 97.7 \times 10^{-12} = 1 + 21.98$$

Hence 23 taps per filter – with a 4x4 MIMO equalizer total of 92 complex taps.

Alternatively and equally valid method is using solver in the calculator for $r = x/\langle\Delta\tau\rangle$ with an initial guess of 3

$$-12 = \log_{10}\left[\frac{4}{\pi}r\exp\left(-\frac{4r^2}{\pi}\right)\right]$$

Gives $r = 4.81$, $x = 100.9$ ps hence $N_{tap} = 1 + 225 \times 10^9 \times 100.9 \times 10^{-12} = 1 + 22.7$
Hence 24 taps per filter giving 96 complex taps.

[20]

e) Assumption 1 – optical fibre operates in the linear regime
rationale: effective area is large which when combined with high dispersion reduces nonlinearities. It is also assumed the power transmitted is relatively low, being the minimum to just achieve the required SNR
Assumption 2 – transceivers approach Shannon capacity
rationale: the capacity 5 bits per symbol but this is a non-integer of 2.5 bits per symbol for each polarization, hence it is reasonable to assume probabilistic constellation shaping is used, which when combined with soft-decision forward error correction approaches the Shannon limit
Assumption 3 – SNR is dominated by ASE compared to transceiver noise
rationale: submarine links use very good transceivers, and the links are so long that ASE becomes the dominant source of noise
Assumption 4 – The WDM are contiguous within the C-band so the optical bandwidth for 20 wavelengths is $B = 20 \times 0.2 = 4\,THz$

From Shannon

$$C = 2B\log_2(1 + SNR)$$

With the 20 wavelengths, $B = 4$ THz and $C = 20\,Tbit/s$

Therefore minimum SNR is

$$SNR_{min} = 2^{C/(2B)} - 1 = 2^{20/8} - 1 = 4.656 = 6.7\ dB$$

but

$$SNR = \frac{P_{out}}{2h\nu n_{sp}(G-1)BN}$$

where $N = 110$ is the number of amplifiers, $G = 15\ dB = 31.62$

$$\begin{aligned} P_{out} &= 4.656 \times 2 \times 1.3 \times 10^{-19} \times 1.5 \times (31.62 - 1) \times 4 \times 10^{12} \times 110 = 24.5 \times 10^{-3}\ W \\ &= 13.9\ dBm \end{aligned}$$

[20]

Question 2

a)

i) Since

$$\frac{dA_1}{dz} = -j\kappa A_2$$

Then

$$\frac{d^2A_1}{dz^2} = -j\kappa\frac{dA_2}{dz} = -j\kappa(-j\kappa A_1) = -\kappa^2 A_1$$

Therefore solution is

$$A_1(z) = \alpha\cos(\kappa z) + \beta\sin(\kappa z)$$

Substituting $z = 0$ gives $A_1(0) = \alpha$ so $A_1(z) = A_1(0)\cos(\kappa z) + \beta\sin(\kappa z)$

But from $\frac{dA_1}{dz} = -j\kappa A_2$ we note

$$A_2(z) = \frac{j}{\kappa}\frac{dA_1}{dz} = \frac{j}{\kappa}[A_1(0)(-\kappa)\sin(\kappa z) + \beta\kappa\cos(\kappa z)] = -jA_1(0)\sin(\kappa z) + j\beta\cos(\kappa z)$$

Therefore $A_2(0) = j\beta$ hence $\beta = -jA_2(0)$

Therefore

$$A_1(z) = A_1(0)\cos(\kappa z) - jA_2(0)\sin(\kappa z)$$

And

$$A_2(z) = -jA_1(0)\sin(\kappa z) + A_1(0)\cos(\kappa z)$$

Comparing with $\mathbf{A}(z) = S(z)\mathbf{A}(0)$ gives

$$\mathbf{A}(z) = \begin{bmatrix} \cos(\kappa z) & -j\sin(\kappa z) \\ -j\sin(\kappa z) & \cos(\kappa z) \end{bmatrix}\mathbf{A}(0)$$

As required

[20]

ii) If $P_1 = |A_1(z)|^2$ and $P_2 = |A_2(z)|^2$ are the powers out of port 1 and 2 respectively with initial conditions $A_1(0) = \sqrt{P_{in}}$, $A_2(0) = 0$ then $P_1 = P_{in}\cos^2(\kappa z)$ and $P_2 = P_{in}\sin^2(\kappa z)$. Hence for $P_1 = P_2$ requires

$$\cos^2(\kappa z) = \sin^2(\kappa z)$$

Hence

$$\kappa z = \frac{\pi}{4} + n\pi$$

Where $n \in \mathbb{Z}$

Hence

$$L_{min} = \frac{\pi}{4\kappa}$$

[15]

b)

i) The scattering matrix associated with the minimum coupling length is

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -j \\ -j & 1 \end{bmatrix}$$

Hence for signal which are coupled they receive a 90 degree phase shift, so tracing the signal through the coupler we see from the top arm we have $A_{in}\exp(j\phi_1)/2$ and from the lower arm $(-j)^2A_{in}\exp(j\phi_2)/2 = -A_{in}\exp(j\phi_2)/2$ hence the overall signal is

$$A_{out} = A_{in}\frac{\exp(j\phi_1) - \exp(j\phi_2)}{2}$$

More generally – on the top arm

$A_{in}\cos(\kappa L_1)\cos(\kappa L_2)\exp(j\phi_1)$ and the lower arm $-A_{in}\sin(\kappa L_1)\sin(\kappa L_2)\exp(j\phi_2)$
Hence the overall response is

$$A_{out} = A_{in}[\cos(\kappa L_1)\cos(\kappa L_2)\exp(j\phi_1) - \sin(\kappa L_1)\sin(\kappa L_2)\exp(j\phi_2)]$$

[20]

ii) Assume minimum length for L_1 and L_2 such that $L_1 = L_2 = \pi/(4\kappa)$
hence

$$A_{out} = \frac{A_{in}}{2}[\exp(j\phi_1) - \exp(j\phi_2)]$$

Substituting expressions for ϕ_1 and ϕ_2 gives

$$A_{out} = \frac{A_{in}}{2}[\exp(j[\phi_0 + V\pi/V_\pi]) - \exp(j[\phi_0 - V\pi/V_\pi])]$$

$$A_{out} = \frac{A_{in}}{2}\exp(j\phi_0)\,[\exp(jV\pi/V_\pi) - \exp(-jV\pi/V_\pi)]$$

$$A_{out} = A_{in} \times j\exp(j\phi_0)\frac{\exp\left(\frac{jV\pi}{V_\pi}\right) - \exp\left(-\frac{jV\pi}{V_\pi}\right)}{2j} = A_{in}j\exp(j\phi_0)\sin\left(\frac{V\pi}{V_\pi}\right)$$

Want $j\exp(j\phi_0) = 1$ therefore $\phi_0 = -\frac{\pi}{2} + 2m\pi$
A suitable value might be therefore $\phi_0 = \frac{3\pi}{2}$ since about this bias point the ϕ_1 and ϕ_2 are positive.

[20]

iii) The phase modulator should be concatenated with the Mach Zehnder modulator (either before or afterwards).
A clear advantage of this approach is that the 3 dB inherent loss associated with the Cartesian modulator is avoided so 3 dB more power is transmitted (and not lost).
The major disadvantage is that the signal processing would become more challenging since it would have to be converted from real and imaginary parts into polar co-ordinates.

[25]

Question 3

a)

i) From the question we have

$$NSR = NSR_0 + \frac{a}{P} + bP^2$$

Hence

$$\frac{dNSR}{dP} = -\frac{a}{P^2} + 2bP$$

Hence at the minima

$$-\frac{a}{P^2} + 2bP = 0$$

So

$$2bP_{opt}^3 = a$$

Hence

$$P_{opt} = \sqrt[3]{\frac{a}{2b}}$$

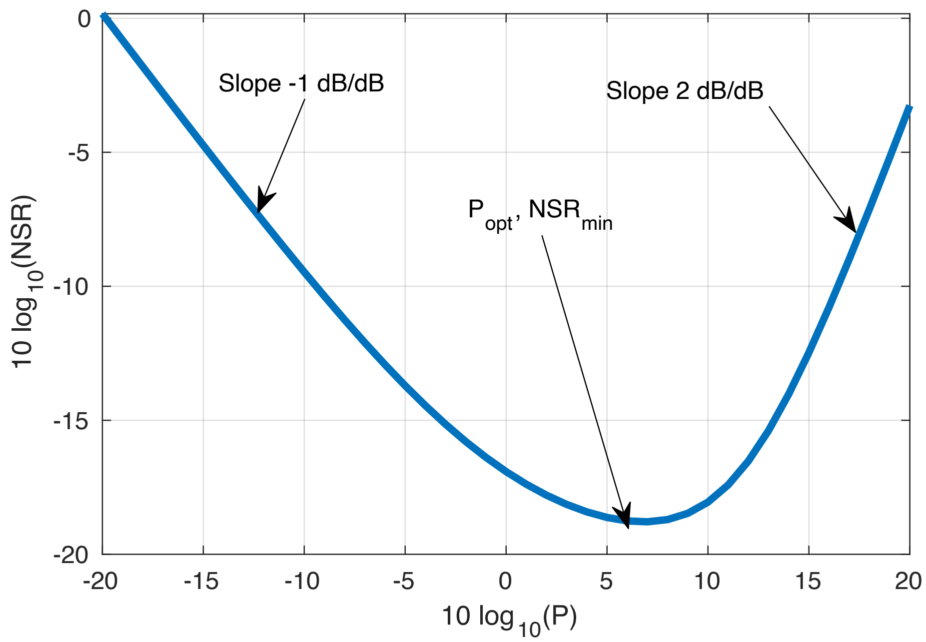
Minimum NSR at this point is

$$NSR_{min} = NSR_0 + \frac{1}{P_{opt}}\left(a + bP_{opt}^3\right) = NSR_0 + \sqrt[3]{\frac{2b}{a}} \times \left(a + b\frac{a}{2b}\right) = NSR_0 + \sqrt[3]{\frac{2b}{a}} \times \frac{3}{2}a$$

$$= NSR_0 + \frac{3}{2} \times \sqrt[3]{2a^2b}$$

[20]

ii)



Key features For $P \ll P_{opt}$, $NSR \approx \frac{a}{P}$

Hence

$$10\log_{10} NSR \sim -10\log_{10} P$$

So NSR falls at a rate of -1 dB per dB of increase in power

Likewise for $P \gg P_{opt}$, $NSR \approx bP^2$

Hence

$$10\log_{10} NSR \sim -20\log_{10} P$$

So NSR increases at a rate of 1 dB per dB of increase in power

[15]

b)

i) Rectangular Nyquist spectrum means that a 200 GBd signal will occupy 200 GHz of optical spectrum and hence 25 channels occupy 5 THz.

$$C_{NLI} = \frac{8\gamma^2 L_{eff}^2 \alpha}{27\pi|\beta_2|} \ln\left(\frac{|\beta_2|}{\alpha}\pi^2 B^2\right)$$

Where $B = 5$ THz hence noting for 100km,

$$L_{eff} \approx \frac{1}{\alpha} = \frac{1}{0.23 \times 0.2} = 21.7 \text{ km}$$

and

$$\beta_2 = -\frac{17}{0.78} = -21.8 \text{ ps}^2/\text{km}$$

hence

$$C_{NLI} = \frac{8 \times 1.3^2 \times 21.7}{27 \times \pi \times 21.8} \ln\left(\frac{21.8}{1/21.7} \times \pi^2 \times 5^2\right) = 1.85\ (pJ)^{-2}$$

$$N_{ASE} = 2n_{sp}h\nu(G-1)$$

$$G = 20\ dB = 100,\ n_{sp} = 2,\ h\nu = 1.3 \times 10^{-19}\ J = 1.3 \times 10^{-7}\ pJ$$

$$N_{ASE} = 2 \times 2 \times 1.3 \times 10^{-7}(100-1) = 5.15 \times 10^{-5}\ pJ$$

Noting that NSR_0 does not change the optimum launch power we use the formula in the formula sheet to give

$$G_{opt} = \sqrt[3]{\frac{N_{ASE}}{2C_{NLI}}} = \sqrt[3]{\frac{5.15 \times 10^{-5}}{2 \times 1.85}} = 0.024\ pJ = 0.024 \frac{\text{mW}}{\text{GHz}}$$

Hence for 200 GHz $P_{opt} = 200 \times 0.024 = 4.8\ mW = 10\log_{10} 4.8 = 6.8\ dBm$

[30]

ii) At the optimum launch power the NSR is given by

$$NSR = NSR_0 + \frac{3}{2} \times \sqrt[3]{2a^2 b}$$

Where $NSR = NSR_0 + \frac{a}{P} + bP^2$

c.f. form in the formula sheet with power spectral density G_{TX} gives

$$NSR = NSR_0 + \frac{N_{ASE}}{G_{TX}} + C_{NLI}G_{TX}^2$$

Hence with 10 spans

$$NSR = 10^{-2} + \frac{10 \times 5.15 \times 10^{-5}}{0.024} + 10 \times 1.85 \times 0.024^2 = 0.042 = -13.8\ dB$$

At this point there is a margin of 1 dB compared to the minimum requirement of -12.8 dB

To determine the range use calculator solver (on fx-991CW home – equation – solver) to solve

$$10^{-2} - 10^{-1.28} + \frac{10 \times 5.15 \times 10^{-5}}{x} + 10 \times 1.85 \times x^2 = 0$$

Solution 1 $x = 0.013$ (initialise at 0.001)
Solution 2 $x = 0.04$ (initialise at 0.1)

This corresponds to

$$P_{lower} = 0.013 \times 200 = 2.6 \text{ mW} = 4.1 \text{ dBm}$$

And

$$P_{upper} = 0.04 \times 200 = 8 \text{ mW} = 9.0 \text{ dBm}$$

So a range of 5 dB.

[35]